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On a singular perturbation in the linear dispersive theory

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Introduction

It is well-known that the initial value problems for the symmetric hyperbolic systems ([9]) or conservation laws ([14]) describe the wave propagation processes, where no dissipation or dispersion of energy takes place. Efficient methods for solving such problems for linear strictly hyperbolic operators asymptotically in the case, when the initial data depend on a small parameter and are highly oscillatory (as the parameter vanishes), were indicated in [15], (see also [1], [11], [12], [17]-[21], and the references there).

Actually, evolution processes in the wave propagation theory are characterized by the loss and (or) dispersion of energy ([27]) and in some situations the coefficients in the dissipative or dispersive terms may be considered as small dimensionless parameters. The Burger (see, for instance, [22]) and the Korteweg-de Vries ([13]) equations are well-known mathematical models, reflecting these phenomena (see [16] as far as the Cauchy problem for the Korteweg-de Vries equation with a small dispersion term is concerned).

In this thesis mixed initial-boundary value problems with coercive boundary operators ([3]) are considered for the linearized Korteweg-de Vries equation affected by the presence of a small dispersive term, and some ideas put forward in [4] are worked out for this specific dispersive singular perturbation. The singularities (in the sense of [2]) of the families of kernels, solving these problems, are investigated as the small parameter, characterizing the dispersion, vanishes, using the saddle-point method and the stationary-phase method. It turns out that the time-dependent singular perturbations similar to the linearized Korteweg-de Vries equation, which is dispersive in the case of the Cauchy problem, may lose this property on a finite interval in the space variable, becoming actually dissipative, the corresponding initial-boundary value problem being well-posed only for the positive values of the time variable.

Special attention is given to the eigenvalue problem for the singular perturbation on a finite interval in the space variable, which generates the singularly perturbed semigroup, associated with the corresponding evolution process (see also [5], [6], where this analysis has been initiated). Yet, for some dispersion terms, the dispersive nature of the corresponding singularly perturbed initial-boundary value problems is preserved on a finite interval (in the space variable) in the following sense: for a sequence of discrete values ϵ_p, $p = 1, 2, \ldots, \epsilon_p \to 0$ as $p \to \infty$, of the small parameter $\epsilon > 0$, which characterizes the level of dispersion, exist purely imaginary simple eigenvalues $\lambda_k(\epsilon_p)$, $1 \leq k \leq n_p$, and, moreover, their number n_p increases to infinity, as $p \to \infty$. Thus a sort of a quantum effect takes place in this case, the quantization being expressed by the discrete values ϵ_p, $p = 1, 2, \ldots$, of the small parameter $\epsilon > 0$. These eigenvalues $\lambda_k(\epsilon_p)$, $1 \leq k \leq n_p$, exhaust the spectrum of the operator in the space variable, which generates the corresponding singularly perturbed semi-group, as $\epsilon_p \to 0$ for $p \to \infty$.

Finally, a specific differential-finite difference approximation of the linear hyperbolic differential operator $\partial_{x_0} + \partial_{x_1}$ is considered, and a similar analysis (investigation of singularities of kernels of the Green and Poisson operators and determination of the eigenvalues of the corresponding eigenvalue problem) is carried through.

This thesis should be considered as a first step towards the investigation of the mixed initial-boundary value problems for linear dispersive singular perturbations of the form $\partial_t + P(x,t,\epsilon,\epsilon\partial_x)\partial_x$ where $P(x,t,0,\partial_x)$ is a strongly elliptic matrix operator ([24]). The final goal is a full understanding of dynamical processes, described by mixed problems for the dissipative and dispersive singular perturbations of the quasi-linear conservation law systems.

The thesis is arranged as follows. First, for the sake of completeness, the Cauchy problem for the linearized Korteweg-de Vries singular perturbation is considered in chapter 1, followed by the analysis of the mixed problem on a half-line in the space variable in chapter 2. In chapter 3, a similar analysis is carried through in the case of the mixed problem, when the space variable belongs to a finite interval. In chapter 4 a detailed asymptotic analysis of the eigenvalue problem for the singular perturbations in the space variable on a finite interval is presented, yielding the conclusions hereabove concerning the dissipative or dispersive nature of the corresponding time-dependent singular perturbations and the associated evolution processes. Finally, in chapter 5, a finite difference approximation is considered.

The results, presented in this thesis, are published in [7], [8].

Chapter 1

The Cauchy problem

This chapter is included in order to make the thesis self-contained. Consider the following singularly perturbed initial value problem:

$$(1.1)\quad (\partial_{x_0} + \partial_{x_1} - \epsilon^2 \partial_{x_1}^3) u_\epsilon(x_0, x_1) = f(x_0, x_1), \qquad x_0 > 0,\ x_1 \in \mathbf{R},$$

$$(1.2)\quad u_\epsilon(0, x_1) = \phi(x_1), \qquad x_1 \in \mathbf{R},$$

where $\partial_{x_k} = \partial/\partial x_k$, $k = 0, 1$, the data f and ϕ are supposed to belong to appropriate spaces and $\epsilon \in (0, \epsilon_0]$ is a given small parameter.

Using the partial Fourier transform $F_{x_1 \to \xi_1}$, one easily finds the fundamental solution of the Cauchy problem for the singular perturbation on the left-hand side of (1.1), i.e. the solution $E_\epsilon(x_0, x_1)$ of (1.1), (1.2) with $f(x_0, x_1) \equiv 0$ and $\phi(x_1) = \delta(x_1)$:

$$(1.3)\qquad E_\epsilon(x_0, x_1) = (2\pi\epsilon)^{-1} \int_{\mathbf{R}} \exp\left(i\epsilon^{-1} h(x_0, x_1, \xi_1)\right) d\xi_1,$$

where

$$h(x_0, x_1, \xi_1) := \xi_1 (x_1 - x_0) - \xi_1^3 x_0.$$

The integral on the right-hand side of (1.3) being convergent in the usual sense for $x_0 \neq 0$, it can also be expressed in terms of the Airy function (see, for instance, [26]).

Using the stationary-phase method (see, for instance, [23]), one finds an asymptotic expansion for E_ϵ in the region $x_1 > x_0$, which exhibits a highly oscillatory behaviour of $E_\epsilon(x_0, x_1)$:

$$(1.4)\quad E_\epsilon(x_0, x_1) \sim 2\,\mathrm{Re}\,\Big(\exp\left(\left(-2/\left(3\sqrt{3}\right)\right) i\epsilon^{-1}(x_1 - x_0)^{3/2} {x_0}^{-1/2}\right) \cdot$$
$$\cdot \sum_{k=0}^{\infty} a_k (x_1 - x_0)^{-(3/2)k - 1/4} {x_0}^{(1/2)k - 1/4} \epsilon^{k - 1/2}\Big)$$
$$\text{as } \epsilon \downarrow 0,$$

where $a_0 = (1/2) \cdot \pi^{-1/2} \cdot 3^{-1/4} \cdot \exp(i\pi/4)$ and $a_k \in \mathbf{C}$, $k > 0$, can be computed explicitly.

For $x_1 < x_0$, $x_0 > 0$ the function $\xi_1 \to (\partial_{\xi_1} h)(x_0, x_1, \xi_1)$ has two zeros on the imaginary axis: $\xi_1^{\pm} = \pm i\Big((3x_0)^{-1}(x_0 - x_1)\Big)^{1/2}$. Writing $\xi_1 = \alpha + i\beta$ we have:

$$\begin{aligned} ih(x_0, x_1, \xi_1) = \; & \beta(x_0 - x_1 + 3\alpha^2 x_0 - \beta^2 x_0) + \\ & + i\alpha(x_1 - x_0 - \alpha^2 x_0 + 3\beta^2 x_0). \end{aligned}$$

Let

$$\begin{aligned} V &:= \{\xi_1 \in \mathbf{C} \mid \operatorname{Re}(ih(x_0, x_1, \xi_1)) < 0\} = \\ &= \{\alpha + i\beta \mid -\left(x_0^{-1}(x_0 - x_1 + 3\alpha^2 x_0)\right)^{1/2} < \beta < 0\} \cup \\ &\cup \{\alpha + i\beta \mid \beta > \left(x_0^{-1}(x_0 - x_1 + 3\alpha^2 x_0)\right)^{1/2}\}. \end{aligned}$$

One has:

$$\operatorname{Im}(ih(x_0, x_1, \xi_1)) = 0,$$

iff

$$\alpha = 0 \text{ or } \beta = \pm\left((3x_0)^{-1}(x_0 - x_1 + \alpha^2 x_0)\right)^{1/2}.$$

Let γ be the curve in the complex plane with the following parametrization:

$$\gamma(t) := t - i\left((3x_0)^{-1}(x_0 - x_1 + t^2 x_0)\right)^{1/2}, \; t \in \mathbf{R}.$$

One easily verifies that γ lies entirely within V; the same is true for every segment $[t; \gamma(t)] := \{\lambda t + (1 - \lambda)\gamma(t) \mid 0 < \lambda < 1\}$, $t \in \mathbf{R}$. Using Cauchy's theorem we find:

$$\begin{aligned} E_\epsilon(x_0, x_1) &= (2\pi\epsilon)^{-1} \int_\gamma \exp\left(i\epsilon^{-1} h(x_0, x_1, \xi_1)\right) d\xi_1 = \\ &= (2\pi\epsilon)^{-1} \int_{\mathbf{R}} \exp\left(i\epsilon^{-1} h(x_0, x_1, \gamma(t))\right) (\partial_t \gamma)(t) dt = \\ &= (2\pi\epsilon)^{-1} \int_{\mathbf{R}} \exp\left(-\epsilon^{-1}\left((3x_0)^{-1}(x_0 - x_1 + t^2 x_0)\right)^{1/2} \cdot \right. \\ &\qquad \cdot \left((2/3)(x_0 - x_1) + (8/3)t^2 x_0\right) dt, \end{aligned}$$

the last formula being a consequence of the fact that the integral over $\mathbf{R}$ of an odd function is zero. Applying the saddle-point method, one finds in the case considered:

$$\begin{aligned} (1.5) \quad E_\epsilon(x_0, x_1) \sim \; & \exp\left(\left(-2/(3\sqrt{3})\right)\epsilon^{-1}(x_0 - x_1)^{3/2} x_0^{-1/2}\right) \cdot \\ & \cdot \sum_{k=0}^{\infty} b_k (x_0 - x_1)^{-(3/2)k - 1/4} x_0^{(1/2)k - 1/4} \epsilon^{k - 1/2} \text{ as } \epsilon \downarrow 0, \end{aligned}$$

where $b_0 = (1/2)\cdot\pi^{-1/2}\cdot 3^{-1/4}$ and $b_k \in \mathbf{R}$, $k > 0$, can be computed explicitly.

Asymptotic expansions for derivatives of E_ϵ are obtained by differentiating formally the expansions (1.4), (1.5). Therefore we have for the singular support of E_ϵ ([2], [4]):

$$\text{(1.6)} \qquad \text{sing supp } E_\epsilon = \{(x_0, x_1) \in \overline{\mathbf{R}_+} \times \mathbf{R} \,|\, x_1 \geq x_0\}.$$

Summarizing we have:

Theorem 1.1 *The fundamental solution E_ϵ of Cauchy problem* (1.1), (1.2) *is given by* (1.3), *its asymptotic expansion by* (1.4) *and* (1.5) *and its singular support by* (1.6).

The solution u_ϵ of (1.1), (1.2) with $f \in L^2(\mathbf{R}_+ \times \mathbf{R}), \phi \in L^2(\mathbf{R})$ is given by

$$u_\epsilon(x_0, x_1) = \int_0^{x_0}\int_{\mathbf{R}} E_\epsilon(x_0 - y_0, x_1 - y_1) f(y_0, y_1) dy_1 dy_0 +$$

$$+ \int_{\mathbf{R}} E_\epsilon(x_0, x_1 - y_1)\phi(y_1) dy_1.$$

In order to get a posteriori estimates we use the following Sobolev spaces: for $m \in \mathbf{N}, \tau > 0$, define (see, for instance, [10]):

$$H_{m,\tau}(\mathbf{R}_+ \times \mathbf{R}) := \{u \in D'(\mathbf{R}_+ \times \mathbf{R}) \,|\, \|u\|_{m,\tau} < \infty\},$$

where the norm $\|\cdot\|_{m,\tau}$ is defined as follows:

$$\|u\|_{m,\tau} := \{\sum_{|\alpha|\leq m} \int_{\mathbf{R}_+}\int_{\mathbf{R}} |\exp(-\tau x_0) D^\alpha u(x_0, x_1)|^2 dx_1 dx_0\}^{1/2}$$

(here $\alpha = (\alpha_0, \alpha_1) \in \mathbf{N}^2$, $|\alpha| = \alpha_0 + \alpha_1$, and $D^\alpha = (-i\partial_{x_0})^{\alpha_0}(-i\partial_{x_1})^{\alpha_1}$).

For $(m_0, m_1) \in \mathbf{N}^2$, $\epsilon > 0$, define the spaces $H_{(m_0,m_1),\epsilon}$ as the completion of $C_0^\infty(\mathbf{R})$ with respect to the norm

$$\|u\|_{(m_0,m_1),\epsilon} = \{\sum_{0\leq k\leq m_0}\sum_{0\leq l\leq m_1} \epsilon^{2l}\int_{\mathbf{R}} |(\partial_{x_1}^{k+l} u)(x_1)|^2 dx_1\}^{1/2}$$

(see [25] where these spaces have been introduced, see also [3] for the definition of the spaces $H_{(s),\epsilon}$ of vectorial order s for every $s \in \mathbf{R}^3$).

Now consider the Cauchy problem (1.1), (1.2) with $f \in C_0^\infty(\mathbf{R}_+ \times \mathbf{R})$ and $\phi = 0$. Substituting $v(x_0, x_1) = \exp(-\tau x_0) u(x_0, x_1)$, $g(x_0, x_1) = \exp(-\tau x_0) f(x_0, x_1)$, where $\tau > 0$, v is a solution of the following Cauchy problem:

$$\text{(1.7)} \quad (\partial_{x_0} + \partial_{x_1} - \epsilon^2\partial_{x_1}^3 + \tau) v(x_0, x_1) = g(x_0, x_1), \quad x_0 > 0, x_1 \in \mathbf{R},$$

$$\text{(1.8)} \quad v(0, x_1) = 0, \qquad\qquad x_1 \in \mathbf{R}.$$

Denoting by $l_0 v, l_0 g$ the extensions by zero for $x_0 < 0$ of v and g respectively, we have, as a consequence of (1.7), (1.8):

$$(1.9)\quad (\partial_{x_0} + \partial_{x_1} - \epsilon^2 \partial_{x_1}^3 + \tau)(l_0 v)(x_0, x_1) = (l_0 g)(x_0, x_1),\ (x_0, x_1) \in \mathbf{R}^2.$$

Application of the Fourier transform to (1.9) yields:

$$(1.10)\quad (i\xi_0 + i\xi_1 - \epsilon^2 (i\xi_1)^3 + \tau)\widehat{l_0 v}(\xi_0, \xi_1) = \widehat{l_0 g}(\xi_0, \xi_1),\ (\xi_0, \xi_1) \in \mathbf{R}^2,$$

where $\hat{u}(\xi_0, \xi_1) = F_{(x_0,x_1)\to(\xi_0,\xi_1)}\left(u\left(x_0, x_1\right)\right)$. Thus (1.10) yields:

$$(1.11)\qquad \widehat{l_0 v}(\xi_0, \xi_1) = \left(i\left(\xi_0 + \xi_1 + \epsilon^2 \xi_1^3\right) + \tau\right)^{-1} \widehat{l_0 g}(\xi_0, \xi_1).$$

Note that $\xi_0 \to \widehat{l_0 v}(\xi_0, \xi_1)$ can be extended analytically for $\operatorname{Im}\xi_0 < 0$, since $\tau > 0$. As a consequence of (1.11) we have:

$$\begin{aligned} \|u\|_{0,\tau} &= \|v\|_{L^2(\mathbf{R}_+\times\mathbf{R})} = \|l_0 v\|_{L^2(\mathbf{R}^2)} \le \tau^{-1}\|l_0 g\|_{L^2(\mathbf{R}^2)} = \\ &= \tau^{-1}\|g\|_{L^2(\mathbf{R}_+\times\mathbf{R})} = \tau^{-1}\|f\|_{0,\tau}. \end{aligned}$$

Since $w(x_0, x_1) = D^\alpha u(x_0, x_1)$ is a solution of (1.1) with right-hand side $D^\alpha f(x_0, x_1)$ and $w(0, x_1) = 0$, we also have:

$$\|D^\alpha u\|_{0,\tau} \le \tau^{-1}\|D^\alpha f\|_{0,\tau} \text{ for all } \alpha \in \mathbf{N} \times \mathbf{N}.$$

Thus one gets the estimate:

$$\|u\|_{m,\tau} \le \tau^{-1}\|f\|_{m,\tau}.$$

Now consider the Cauchy problem (1.1), (1.2) with $f = 0$ and $\phi \in C_0^\infty(\mathbf{R})$. After the partial Fourier transform $F_{x_1\to\xi_1}$ we find for $\tilde{u}(x_0, \xi_1) := F_{x_1\to\xi_1}\left(u(x_0, x_1)\right)$ the following formula:

$$\tilde{u}(x_0, \xi_1) = \hat{\phi}(\xi_1)\exp(-i\xi_1(1 + \epsilon^2\xi_1^2)x_0),$$

where $\hat{\phi}(\xi_1) = F_{x_1\to\xi_1}\left(\phi(x_1)\right)$. For $(\alpha, \beta) \in \mathbf{N} \times \mathbf{N}$ we have:

$$\widetilde{\partial_{x_0}^\alpha \partial_{x_1}^\beta u}(x_0, \xi_1) = (-1)^\alpha (i\xi_1)^{\alpha+\beta}(1 + \epsilon^2\xi_1^2)^\alpha \hat{\phi}(\xi_1)\exp(-i\xi_1(1 + \epsilon^2\xi_1^2)x_0).$$

Therefore one gets for $\tau > 0$:

$$\int_0^\infty \int_{\mathbf{R}} |(\partial_{x_0}^\alpha \partial_{x_1}^\beta u)(x_0, x_1)|^2 \exp(-2\tau x_0) dx_1 dx_0 =$$

$$= (2\pi)^{-1}\int_0^\infty \int_{\mathbf{R}} |(\widetilde{\partial_{x_0}^\alpha \partial_{x_1}^\beta u})(x_0,\xi_1)|^2 \exp(-2\tau x_0) d\xi_1 dx_0 =$$

$$= (2\pi)^{-1}\int_0^\infty \int_{\mathbf{R}} \xi_1^{2(\alpha+\beta)}(1+\epsilon^2\xi_1^2)^{2\alpha}|\hat{\phi}(\xi_1)|^2 \exp(-2\tau x_0) d\xi_1 dx_0 =$$

$$= (2\pi)^{-1}(2\tau)^{-1}\int_{\mathbf{R}} \xi_1^{2(\alpha+\beta)}(1+\epsilon^2\xi_1^2)^{2\alpha}|\hat{\phi}(\xi_1)|^2 d\xi_1 =$$

$$= (2\tau)^{-1}\sum_{k=0}^{2\alpha}\begin{pmatrix} 2\alpha \\ k \end{pmatrix}\epsilon^{2k}\int_{\mathbf{R}} |(\partial_{x_1}^{\alpha+\beta+k}\phi)(x_1)|^2 dx_1.$$

Thus one has:

$$\|\partial_{x_0}^\alpha \partial_{x_1}^\beta u\|_{0,\tau}^2 = (2\tau)^{-1}\sum_{k=0}^{2\alpha}\begin{pmatrix} 2\alpha \\ k \end{pmatrix}\epsilon^{2k}\|\partial_{x_1}^{\alpha+\beta+k}\phi\|_{L^2(\mathbf{R})}^2.$$

Taking the sum over all $(\alpha,\beta) \in \mathbf{N}\times\mathbf{N}$ with $\alpha+\beta \le m$ we get:

$$\text{(1.12)} \qquad \|u\|_{m,\tau}^2 = (2\tau)^{-1}\sum_{\alpha+\beta\le m}\sum_{k=0}^{2\alpha}\begin{pmatrix} 2\alpha \\ k \end{pmatrix}\epsilon^{2k}\|\partial_{x_1}^{\alpha+\beta+k}\phi\|_{L^2(\mathbf{R})}^2.$$

One easily verifies that there exist constants $C_1 > 0, C_2 > 0$, which do not depend on $\phi \in C_0^\infty(\mathbf{R})$, $\epsilon \in (0,1]$ and are such that

$$\text{(1.13)} \quad C_1\|\phi\|_{(m,2m),\epsilon}^2 \le \sum_{\alpha+\beta\le m}\sum_{k=0}^{2\alpha}\begin{pmatrix} 2\alpha \\ k \end{pmatrix}\epsilon^{2k}\|\partial_{x_1}^{\alpha+\beta+k}\phi\|_{L^2(\mathbf{R})}^2 \le$$
$$\le C_2\|\phi\|_{(m,2m),\epsilon}^2.$$

Combining (1.12) and (1.13) we conclude that there exists a constant $C > 0$, which does not depend on $\phi \in C_0^\infty(\mathbf{R})$, $\epsilon \in (0,1]$, $\tau > 0$ and is such that the following two-sided estimate holds:

$$C^{-1}\|\phi\|_{(m,2m),\epsilon} \le \tau^{1/2}\|u\|_{m,\tau} \le C\|\phi\|_{(m,2m),\epsilon}.$$

We have proved the following result:

Theorem 1.2 *For the solution u of the Cauchy problem* (1.1), (1.2) *with $f \in C_0^\infty(\mathbf{R}_+ \times \mathbf{R})$, $\phi \in C_0^\infty(\mathbf{R})$ the following estimate holds:*

$$\|u\|_{m,\tau} \le C_1\{\tau^{-1}\|f\|_{m,\tau} + \tau^{-1/2}\|\phi\|_{(m,2m),\epsilon}\},$$

where the constant $C_1 > 0$ does not depend on f, ϕ, $\tau > 0$ and $\epsilon \in (0,1]$. If $f = 0$ one also has:

$$\|u\|_{m,\tau} \ge C_2\tau^{-1/2}\|\phi\|_{(m,2m),\epsilon},$$

where the constant $C_2 > 0$ does not depend on ϕ, $\tau > 0$ and $\epsilon \in (0,1]$.

Next we show the following convergence result:

Theorem 1.3 *Assume that*

$$f \in L^1_{loc}\left([0,\infty), H^3(\mathbf{R})\right), \phi \in H^3(\mathbf{R}).$$

For u_ϵ the solution of the Cauchy problem (1.1), (1.2) *and u_0 the solution of the corresponding reduced problem with $\epsilon = 0$ the following estimate holds uniformly with respect to $\epsilon > 0$:*

$$\|u_\epsilon(x_0,\cdot) - u_0(x_0,\cdot)\|_{L^2(\mathbf{R}_{x_1})} \le K\epsilon^2,$$

where the constant $K > 0$ may depend on f, ϕ and x_0, but not on ϵ.

Proof. The partial Fourier transform $F_{x_1\to\xi_1}$ yields:

$$\begin{aligned}\tilde{u}_\epsilon(x_0,\xi_1) = \int_0^{x_0} & \exp\left(i\xi_1(1+\epsilon^2\xi_1^2)(y_0 - x_0)\right)\tilde{f}(y_0,\xi_1)dy_0 + \\ & +\hat{\phi}(\xi_1)\exp\left(-i\xi_1(1+\epsilon^2\xi_1^2)x_0\right), \ x_0 > 0, \ \xi_1 \in \mathbf{R},\end{aligned}$$

$$\begin{aligned}\tilde{u}_0(x_0,\xi_1) = \int_0^{x_0} & \exp\left(i\xi_1(y_0 - x_0)\right)\tilde{f}(y_0,\xi_1)dy_0 + \\ & +\hat{\phi}(\xi_1)\exp\left(-i\xi_1 x_0\right), \ x_0 > 0, \ \xi_1 \in \mathbf{R},\end{aligned}$$

where $\tilde{u}(x_0,\xi_1) = F_{x_1\to\xi_1}\left(u(x_0,x_1)\right), \hat{v}(\xi_1) = F_{x_1\to\xi_1}\left(v(x_1)\right)$.
Now we have:

$$\|u_\epsilon(x_0,\cdot) - u_0(x_0,\cdot)\|_{L^2(\mathbf{R}_{x_1})} =$$

$$= (2\pi)^{-1/2}\|\tilde{u}_\epsilon(x_0,\cdot) - \tilde{u}_0(x_0,\cdot)\|_{L^2(\mathbf{R}_{\xi_1})} \le$$

$$\begin{aligned}\le (2\pi)^{-1/2}\Big\|\int_0^{x_0} & \left(\exp\left(i\xi_1(1+\epsilon^2\xi_1^2)(y_0 - x_0)\right) + \right. \\ & \left. - \exp\left(i\xi_1(y_0 - x_0)\right)\right)\tilde{f}(y_0,\xi_1)dy_0\Big\|_{L^2(\mathbf{R}_{\xi_1})} +\end{aligned}$$

$$+(2\pi)^{-1/2}\|\left(\exp\left(-i\xi_1(1+\epsilon^2\xi_1^2)x_0\right) - \exp\left(-i\xi_1 x_0\right)\right)\hat{\phi}(\xi_1)\|_{L^2(\mathbf{R}_{\xi_1})} \le$$

$$\begin{aligned}\le (2\pi)^{-1/2}\int_0^{x_0} \Big\| & \left(\exp\left(i\xi_1(1+\epsilon^2\xi_1^2)(y_0 - x_0)\right) + \right. \\ & \left. - \exp\left(i\xi_1(y_0 - x_0)\right)\right)\tilde{f}(y_0,\xi_1)\Big\|_{L^2(\mathbf{R}_{\xi_1})}dy_0 +\end{aligned}$$

$$+(2\pi)^{-1/2}\|\left(\exp\left(-i\xi_1(1+\epsilon^2\xi_1^2)x_0\right) - \exp\left(-i\xi_1 x_0\right)\right)\hat{\phi}(\xi_1)\|_{L^2(\mathbf{R}_{\xi_1})} =$$

$$= (2\pi)^{-1/2}\epsilon^2 \int_0^{x_0} \| \ (\exp(i\xi_1(1+\epsilon^2\xi_1^2)(y_0-x_0)) - \exp(i\xi_1(y_0-x_0)))\,\cdot$$

$$\cdot\,(i\epsilon^2\xi_1^3(y_0-x_0))^{-1}\cdot i\xi_1^3(y_0-x_0)\tilde{f}(y_0,\xi_1)\|_{L^2(\mathbf{R}_{\xi_1})}dy_0 +$$

$$+(2\pi)^{-1/2}\epsilon^2 \ \| (\exp(-i\xi_1(1+\epsilon^2\xi_1^2)x_0) - \exp(-i\xi_1 x_0))\,\cdot$$

$$\cdot(-i\epsilon^2\xi_1^3 x_0)^{-1}\cdot -i\xi_1^3 x_0\hat{\phi}(\xi_1)\|_{L^2(\mathbf{R}_{\xi_1})} \leq$$

$$\leq \epsilon^2\left(\int_0^{x_0}(x_0-y_0)\|(\partial_{x_1}^3 f)(y_0,\cdot)\|_{L^2(\mathbf{R}_{x_1})}dy_0 + x_0\|\partial_{x_1}^3\phi\|_{L^2(\mathbf{R}_{x_1})}\right).$$

For the last inequality we used the fact that $|x^{-1}(\exp(ix)-1)| \leq 1, \forall x \in \mathbf{R}$.

□

Remark. For the singular perturbation

$$\partial_{x_0} + \partial_{x_1} + \epsilon^2\partial_{x_1}^3 \tag{1.14}$$

the fundamental solution $E_\epsilon^-(x_0,x_1)$ has as its singular support (in the sense of [2]) the set $\{(x_0,x_1) \in \mathbf{R}^2 \,|\, x_0(x_1-x_0) \leq 0\}$ and has the asymptotic behaviour similar to (1.4) and (1.5) for $x_0(x_1-x_0) < 0$ and $x_0(x_1-x_0) > 0$ respectively (see also [4], [5] and Example 3.8.15 in [6]). Thus from this point of view there is not much difference between E_ϵ and E_ϵ^- as far as the Cauchy problem is concerned.

Chapter 2

The mixed problem on the half-line

Consider the following singularly perturbed initial-boundary value problem:

$$(2.1)\quad (\partial_{x_0} + \partial_{x_1} - \epsilon^2 \partial_{x_1}^3) u(x_0, x_1) = f(x_0, x_1), \qquad x_0 > 0,\ x_1 > 0,$$

$$(2.2)\quad u(0, x_1) = \phi(x_1), \qquad x_1 > 0,$$

$$(2.3)\quad u(x_0, 0) = \psi_1(x_0), \qquad x_0 > 0,$$

$$(2.4)\quad (\partial_{x_1} u)(x_0, 0) = \psi_2(x_0), \qquad x_0 > 0.$$

We write the solution u_ϵ of (2.1)-(2.4) in the following form:

$$u_\epsilon = G_\epsilon f + H_\epsilon \phi + P_\epsilon^1 \psi_1 + P_\epsilon^2 \psi_2,$$

where G_ϵ, H_ϵ, P_ϵ^1 and P_ϵ^2 are integral operators. In this chapter we are going to determine the kernels of these operators, their asymptotic expansions as $\epsilon \downarrow 0$ and their singular supports.

Consider the equation:

$$(2.5)\qquad i\xi_0 + X - \epsilon^2 X^3 = 0,$$

where $\epsilon > 0$, $\xi_0 \in \mathbf{R}$. Equation (2.5) has three roots $\mu_j(\epsilon, \xi_0)\,(1 \le j \le 3)$ where $\mathrm{Re}\,(\mu_1(\epsilon, \xi_0)) = 0$, $\mathrm{Re}\,(\mu_2(\epsilon, \xi_0)) < 0$ and $\mathrm{Re}\,(\mu_3(\epsilon, \xi_0)) > 0$. Multiplying equation (2.5) by ϵ yields:

$$\epsilon \mu_j(\epsilon, \xi_0) = \mu_j(1, \epsilon\xi_0), \quad 1 \le j \le 3,\ \epsilon > 0,\ \xi_0 \in \mathbf{R},$$

or equivalently

$$\mu_j(\epsilon, \xi_0) = \epsilon^{-1} \mu_j(1, \epsilon\xi_0), \quad 1 \le j \le 3,\ \epsilon > 0,\ \xi_0 \in \mathbf{R}.$$

We will write $\mu_j(\xi_0)$ instead of $\mu_j(1,\xi_0)$.

First we are going to determine the kernel of the operator P_ϵ^1 and its asymptotic expansion as $\epsilon \downarrow 0$. This kernel is also denoted by P_ϵ^1. Therefore consider problem (2.1)-(2.4) with $f = 0$, $\phi = 0$, $\psi_1(x_0) = \delta(x_0), \psi_2 = 0$. Extending the solution P_ϵ^1 by zero for $x_0 < 0$ the extension $l_0 P_\epsilon^1$ is a solution of the following Cauchy problem:

$$
\begin{aligned}
&(\partial_{x_0} + \partial_{x_1} - \epsilon^2 \partial_{x_1}^3)(l_0 P_\epsilon^1)(x_0, x_1) = 0, && x_0 \in \mathbf{R},\ x_1 > 0,\\
&(l_0 P_\epsilon^1)(x_0, 0) = \delta(x_0), && x_0 \in \mathbf{R},\\
&(\partial_{x_1}(l_0 P_\epsilon^1))(x_0, 0) = 0, && x_0 \in \mathbf{R}.
\end{aligned}
$$

After the partial Fourier transform $F_{x_0 \to \xi_0}$ the function

$$x_1 \to \left(F_{x_0 \to \xi_0}(l_0 P_\epsilon^1)\right)(\xi_0, x_1)$$

is found to satisfy a third-order linear ordinary differential equation with two prescribed boundary values at $x_1 = 0$. Since the characteristic equation of this differential equation is (2.5), which has two zeros with non-positive real part and one zero with a positive real part, there is a unique tempered distribution solution of this boundary value problem. After the inverse Fourier transform $F_{\xi_0 \to x_0}^{-1}$ we get for P_ϵ^1 the following formula:

$$
\begin{aligned}
(2.6)\quad & P_\epsilon^1(x_0, x_1) = (2\pi\epsilon)^{-1} \cdot \\
& \cdot \left(\int_{\mathbf{R}} \mu_2(\xi_0) \left(\mu_2(\xi_0) - \mu_1(\xi_0)\right)^{-1} \exp\left(\epsilon^{-1}(\mu_1(\xi_0) x_1 + i\xi_0 x_0)\right) d\xi_0 + \right.\\
& \left. - \int_{\mathbf{R}} \mu_1(\xi_0) \left(\mu_2(\xi_0) - \mu_1(\xi_0)\right)^{-1} \exp\left(\epsilon^{-1}(\mu_2(\xi_0) x_1 + i\xi_0 x_0)\right) d\xi_0 \right),\\
& x_0 > 0,\ x_1 > 0.
\end{aligned}
$$

The corresponding operator is defined by

$$(P_\epsilon^1 \psi_1)(x_0, x_1) = \int_0^{x_0} P_\epsilon^1(x_0 - y_0, x_1) \psi_1(y_0) dy_0.$$

Now we are going to determine the asymptotic expansion of $P_\epsilon^1(x_0, x_1)$. We write

$$P_\epsilon^1(x_0, x_1) = R_\epsilon^1(x_0, x_1) + S_\epsilon^1(x_0, x_1),$$

where

$$R_\epsilon^1(x_0,x_1) =$$

$$= (2\pi\epsilon)^{-1}\int_{\mathbf{R}} \mu_2(\xi_0)\,(\mu_2(\xi_0)-\mu_1(\xi_0))^{-1}\exp\left(\epsilon^{-1}(\mu_1(\xi_0)x_1 + i\xi_0 x_0)\right)d\xi_0,$$

$$S_\epsilon^1(x_0,x_1) =$$

$$= -(2\pi\epsilon)^{-1}\int_{\mathbf{R}} \mu_1(\xi_0)\,(\mu_2(\xi_0)-\mu_1(\xi_0))^{-1}\exp\left(\epsilon^{-1}(\mu_2(\xi_0)x_1 + i\xi_0 x_0)\right)d\xi_0.$$

Using the substitution $\xi_0 = -\alpha-\alpha^3$ and observing that $\mu_1(\xi_0) = i\alpha, \mu_2(\xi_0) = -(1/2)(i\alpha + (3\alpha^2+4)^{1/2})$, one finds:

$$(2.7)\quad R_\epsilon^1(x_0,x_1) =$$

$$\begin{aligned} &= (2\pi\epsilon)^{-1}\int_{\mathbf{R}} \mu_2(\xi_0)(\mu_2(\xi_0)-\mu_1(\xi_0))^{-1}\cdot \\ &\qquad \cdot\exp\left(\epsilon^{-1}\left(\mu_1(\xi_0)x_1 + i\xi_0x_0\right)\right)d\xi_0 = \\ &= (2\pi\epsilon)^{-1}\int_{\mathbf{R}} \left((3\alpha^2+4)^{1/2}+i\alpha\right)\left((3\alpha^2+4)^{1/2}+3i\alpha\right)^{-1}\cdot \\ &\qquad \cdot(1+3\alpha^2)\exp\left(i\epsilon^{-1}\left(\alpha(x_1-x_0)-\alpha^3x_0\right)\right)d\alpha. \end{aligned}$$

Note that the phase function in the second integral on the right-hand side of (2.7) is the same as the one in the integral on the right-hand side of (1.3). Therefore using the same argument as in chapter 1, one gets for R_ϵ^1 the following asymptotic expansion as $\epsilon \downarrow 0$:

$$(2.8)\quad R_\epsilon^1(x_0,x_1) \sim \begin{cases} 2\mathrm{Re}\ \left(\exp\left(\left(-2/(3\sqrt{3})\right)i\epsilon^{-1}(x_1-x_0)^{3/2}x_0^{-1/2}\right)\cdot\right. \\ \qquad \left.\cdot\sum_{n=0}^{\infty}a_n(x_0,x_1)\epsilon^{n-1/2}\right) \text{ for } x_0 < x_1, \\ \exp\ \left(\left(-2/(3\sqrt{3})\right)\epsilon^{-1}(x_0-x_1)^{3/2}x_0^{-1/2}\right)\cdot \\ \qquad \cdot\sum_{n=0}^{\infty}b_n(x_0,x_1)\epsilon^{n-1/2} \text{ for } x_0 > x_1, \end{cases}$$

where

$$a_0(x_0,x_1) = (1/2)\cdot 3^{-1/4}\cdot\exp(i\pi/4)\cdot\pi^{-1/2}\cdot x_1\cdot x_0^{-5/4}\cdot(x_1-x_0)^{-1/4}\cdot$$

$$\cdot[3^{-1/2}i(x_1-x_0)^{1/2}-(3x_0+x_1)^{1/2}][3^{1/2}i(x_1-x_0)^{1/2}-(3x_0+x_1)^{1/2}]^{-1}$$

$$b_0(x_0,x_1) = (1/2)\cdot 3^{-1/4}\cdot\pi^{-1/2}\cdot x_1\cdot x_0^{-5/4}\cdot(x_0-x_1)^{-1/4}\cdot$$

$$\cdot[(3x_0+x_1)^{1/2}+3^{-1/2}(x_0-x_1)^{1/2}][(3x_0+x_1)^{1/2}+3^{1/2}(x_0-x_1)^{1/2}]^{-1}.$$

It will be shown that $S^1_\epsilon(x_0,x_1)$ has no contribution to the asymptotic expansion of $P^1_\epsilon(x_0,x_1)$. In order to do so we observe that equation (2.5) with $\epsilon = 1$ has no multiple roots for $\xi_0 \neq \pm 2i/(3\sqrt{3})$. Therefore $\xi_0 \to \mu_2(\xi_0)$ is an analytic function for $|\mathrm{Im}\,\xi_0| < 2/(3\sqrt{3})$ and we may write:

$$
\begin{aligned}
(2.9)\qquad & S^1_\epsilon(x_0,x_1) = \\
&= -(2\pi\epsilon)^{-1}\int_{\mathbf{R}} \mu_1(\xi_0)(\mu_2(\xi_0)-\mu_1(\xi_0))^{-1}\cdot \\
&\qquad \cdot \exp\left(\epsilon^{-1}\left(\mu_2(\xi_0)x_1 + i\xi_0 x_0\right)\right) d\xi_0 = \\
&= -(2\pi\epsilon)^{-1}\int_{-\infty+i\tau}^{\infty+i\tau} \mu_1(\xi_0)(\mu_2(\xi_0)-\mu_1(\xi_0))^{-1}\cdot \\
&\qquad \cdot \exp\left(\epsilon^{-1}\left(\mu_2(\xi_0)x_1 + i\xi_0 x_0\right)\right) d\xi_0,
\end{aligned}
$$

where $0 < \tau < 2/(3\sqrt{3})$, since $\lim\limits_{|T|\to\infty}\max\limits_{0\le t\le\tau}\mathrm{Re}(\mu_2(T+it)) = -\infty$. With $\xi_0 = \eta + i\tau$ (2.9) becomes:

$$
\begin{aligned}
& S^1_\epsilon(x_0,x_1) = \\
&= (2\pi\epsilon)^{-1}\exp(-\epsilon^{-1}\tau x_0)\int_{\mathbf{R}} \mu_1(\eta+i\tau)\left(\mu_1(\eta+i\tau)-\mu_2(\eta+i\tau)\right)^{-1}\cdot \\
&\qquad \cdot \exp\left(\epsilon^{-1}\left(\mu_2(\eta+i\tau)x_1 + i\eta x_0\right)\right) d\eta.
\end{aligned}
$$

Since $\mathrm{Re}\left(\mu_2(\eta+i\tau)\right) < 0, \forall\eta \in \mathbf{R}$ and $\lim\limits_{|\eta|\to\infty}\mathrm{Re}\left(\mu_2(\eta+i\tau)\right) = -\infty$, one finds:

$$
S^1_\epsilon(x_0,x_1) = O\left(\epsilon^{-1}\exp(-\epsilon^{-1}\tau x_0)\right) \text{ as } \epsilon \downarrow 0.
$$

Noticing that for $x_0 > x_1 > 0$ one has:

$$
\left(2/(3\sqrt{3})\right)(x_0 - x_1)^{3/2}x_0^{-1/2} < \tau x_0,
$$

choosing τ sufficiently close to $2/(3\sqrt{3})$ and using the last asymptotic relation for S^1_ϵ, one comes to the conclusion that $P^1_\epsilon(x_0,x_1)$ has the same asymptotic expansion as $R^1_\epsilon(x_0,x_1)$ given by (2.8). Furthermore one has again:

$$
(2.10)\qquad \text{sing supp}P^1_\epsilon = \{(x_0,x_1) \in \overline{\mathbf{R}_+}\times\overline{\mathbf{R}_+}\,|\,x_0 \le x_1\}.
$$

The operator P^2_ϵ is defined by

$$
(P^2_\epsilon\psi_2)(x_0,x_1) = \int_0^{x_0} P^2_\epsilon(x_0-y_0,x_1)\psi_2(y_0)dy_0,
$$

where $P_\epsilon^2(x_0,x_1)$ is the solution of (2.1)-(2.4) with $f = 0, \phi = 0, \psi_1 = 0$, $\psi_2(x_0) = \delta(x_0)$. Proceeding in the same way as we did in order to determine $P_\epsilon^1(x_0,x_1)$, we find:

$$(2.11) \qquad P_\epsilon^2(x_0,x_1) =$$

$$= (2\pi)^{-1}\int_{\mathbf{R}} (\mu_1(\xi_0)-\mu_2(\xi_0))^{-1} \exp\left(\epsilon^{-1}\left(\mu_1(\xi_0)x_1 + i\xi_0 x_0\right)\right) d\xi_0 +$$
$$-(2\pi)^{-1}\int_{\mathbf{R}} (\mu_1(\xi_0)-\mu_2(\xi_0))^{-1} \exp\left(\epsilon^{-1}\left(\mu_2(\xi_0)x_1 + i\xi_0 x_0\right)\right) d\xi_0$$

and the following asymptotic expansion:

$$(2.12)\ P_\epsilon^2(x_0,x_1) \sim \begin{cases} 2\,\mathrm{Re}\Big(\exp\left(\left(-2/(3\sqrt{3})\right) i\epsilon^{-1}(x_1-x_0)^{3/2}x_0^{-1/2}\right)\cdot \\ \qquad \cdot\displaystyle\sum_{n=0}^{\infty} c_n(x_0,x_1)\epsilon^{n+1/2}\Big) \text{ for } x_1 > x_0 \\ \exp\left(\left(-2/(3\sqrt{3})\right)\epsilon^{-1}(x_0-x_1)^{3/2}x_0^{-1/2}\right)\cdot \\ \qquad \cdot\displaystyle\sum_{n=0}^{\infty} d_n(x_0,x_1)\epsilon^{n+1/2} \text{ for } x_1 < x_0, \end{cases}$$

where

$$\begin{aligned} c_0(x_0,x_1) = {} & 3^{-1/4}\cdot\exp(i\pi/4)\cdot\pi^{-1/2}\cdot x_1\cdot x_0^{-3/4}\cdot(x_1-x_0)^{-1/4}\cdot \\ & \cdot\left[-3^{1/2}i(x_1-x_0)^{1/2}+(3x_0+x_1)^{1/2}\right]^{-1}, \\ d_0(x_0,x_1) = {} & 3^{-1/4}\cdot\pi^{-1/2}\cdot x_1\cdot x_0^{-3/4}\cdot(x_0-x_1)^{-1/4}\cdot \\ & \cdot\left[3^{1/2}(x_0-x_1)^{1/2}+(3x_0+x_1)^{1/2}\right]^{-1}. \end{aligned}$$

Again one has:

$$(2.13) \qquad \text{sing supp } P_\epsilon^2 = \{(x_0,x_1)\in\overline{\mathbf{R}_+}\times\overline{\mathbf{R}_+} \mid x_0 \le x_1\}.$$

The operator H_ϵ is defined by

$$(H_\epsilon\phi)(x_0,x_1) = \int_0^\infty H_\epsilon(x_0,x_1,y_1)\phi(y_1)dy_1,$$

where $H_\epsilon(x_0,x_1,y_1)$ is the solution of (2.1)-(2.4) with $f = 0$, $\phi(x_1) = \delta(x_1-y_1)$, $\psi_1 = 0$, $\psi_2 = 0$. Setting

$$H_\epsilon(x_0,x_1,y_1) = E_\epsilon(x_0,x_1-y_1) + \nu_\epsilon(x_0,x_1,y_1)$$

where E_ϵ is the fundamental solution given by (1.3), ν_ϵ is found to satisfy (2.1)-(2.4) with $f = 0, \phi = 0, \psi_1(x_0) = -E_\epsilon(x_0, -y_1), \psi_2(x_0) = -(\partial_{x_1} E_\epsilon)(x_0, -y_1)$. Therefore

$$\begin{aligned} \nu_\epsilon(x_0, x_1, y_1) = & -\int_0^{x_0} P_\epsilon^1(x_0 - y_0, x_1) E_\epsilon(y_0, -y_1) dy_0 + \\ & -\int_0^{x_0} P_\epsilon^2(x_0 - y_0, x_1)(\partial_{x_1} E_\epsilon)(y_0, -y_1) dy_0. \end{aligned}$$

Thus:

$$\begin{aligned} (2.14) \quad H_\epsilon(x_0, x_1, y_1) = & \; E_\epsilon(x_0, x_1 - y_1) + \\ & -\int_0^{x_0} P_\epsilon^1(x_0 - y_0, x_1) E_\epsilon(y_0, -y_1) dy_0 + \\ & -\int_0^{x_0} P_\epsilon^2(x_0 - y_0, x_1)(\partial_{x_1} E_\epsilon)(y_0, -y_1) dy_0. \end{aligned}$$

In order to get an asymptotic expansion of H_ϵ we have to use (1.4), (1.5), (2.8), (2.12) and the asymptotic expansion of $\partial_{x_1} E_\epsilon$, which is similar to (1.4), (1.5). If $x_1 - y_1 > x_0$ then the asymptotic behaviour of $E_\epsilon(x_0, x_1 - y_1)$ as $\epsilon \downarrow 0$ is oscillatory, while the integrals on the right-hand side of (2.14) are exponentially small, since $E_\epsilon(y_0, -y_1)$ and $(\partial_{x_1} E_\epsilon)(y_0, -y_1)$ are exponentially small. If $x_1 - y_1 < x_0$ then $E_\epsilon(x_0, x_1 - y_1)$ is exponentially small as $\epsilon \downarrow 0$, but the integrals on the right-hand side of (2.14) are asymptotically negligible compared to $E_\epsilon(x_0, x_1 - y_1)$, given that the following inequalities hold:

$$(x_0 - x_1 + y_1)^{3/2} x_0^{-1/2} < (x_0 - y_0 - x_1)^{3/2}(x_0 - y_0)^{-1/2} + (y_0 + y_1)^{3/2} y_0^{-1/2}$$

for $0 < y_0 < x_0 - x_1$,

$$(x_0 - x_1 + y_1)^{3/2} x_0^{-1/2} < (y_0 + y_1)^{3/2} y_0^{-1/2} \text{ for } x_0 - x_1 \leq y_0 \leq x_0.$$

Indeed, the second of the inequalities is obvious, while the first one is readily obtained using the following inequality:

$$(2.15) \quad (\alpha - \beta)^{3/2}(t - \beta)^{-1/2} + (\beta + \gamma)^{3/2}\beta^{-1/2} - (\alpha + \gamma)^{3/2} t^{-1/2} > 0,$$

for $\gamma > 0$, $0 < \beta < \alpha < t$.

Inequality (2.15) is shown as follows.
Introduce:

$$f(t) := (\alpha - \beta)^{3/2}(t - \beta)^{-1/2} + (\beta + \gamma)^{3/2}\beta^{-1/2} - (\alpha + \gamma)^{3/2} t^{-1/2} \text{ for } t > \alpha.$$

Then:

$$\begin{aligned}
&(\partial_t f)(t) = \\
&= -(1/2)(\alpha-\beta)^{3/2}(t-\beta)^{-3/2} + (1/2)(\alpha+\gamma)^{3/2}t^{-3/2} = \\
&= (1/2)t^{-3/2}(t-\beta)^{-3/2}\{-t^{3/2}(\alpha-\beta)^{3/2} + (t-\beta)^{3/2}(\alpha+\gamma)^{3/2}\} = \\
&= (1/2)t^{-3/2}(t-\beta)^{-3/2}\{-(t\alpha-t\beta)^{3/2} + (t\alpha+t\gamma-\alpha\beta-\beta\gamma)^{3/2}\} > \\
&> 0,
\end{aligned}$$

since $(t\alpha + t\gamma - \alpha\beta - \beta\gamma) - (t\alpha - t\beta) = \gamma(t-\beta) + \beta(t-\alpha) > 0$.
Furthermore one has:

$$\lim_{t\downarrow\alpha} f(t) = \alpha - \beta + (\beta+\gamma)^{3/2}\beta^{-1/2} - (\alpha+\gamma)^{3/2}\alpha^{-1/2}.$$

Let $g(\gamma) := \alpha - \beta + (\beta+\gamma)^{3/2}\beta^{-1/2} - (\alpha+\gamma)^{3/2}\alpha^{-1/2}, \gamma > 0$.
Then one has: $f(t) > g(\gamma), \forall t > \alpha$.
One also has:

$$\begin{aligned}
(\partial_\gamma g)(\gamma) &= (3/2)(\beta+\gamma)^{1/2}\beta^{-1/2} - (3/2)(\alpha+\gamma)^{1/2}\alpha^{-1/2} = \\
&= (3/2)\{(1+\gamma\beta^{-1})^{1/2} - (1+\gamma\alpha^{-1})^{1/2}\} > 0.
\end{aligned}$$

Furthermore it is easy to see that $\lim_{\gamma\downarrow 0} g(\gamma) = 0$. Therefore $g(\gamma) > 0, \forall\gamma > 0$, so that $f(t) > 0, \forall t > \alpha$, which proves (2.15).
Thus one reaches the conclusion that $H_\epsilon(x_0, x_1, y_1)$ has the same asymptotic expansion as $E_\epsilon(x_0, x_1 - y_1)$:

(2.16) $\qquad H_\epsilon(x_0, x_1, y_1) \sim$

$$\sim \begin{cases}
2\,\mathrm{Re}\left(\exp\left(i\epsilon^{-1}\phi^+(x_0, x_1 - y_1)\right)\sum_{k=0}^{\infty}\psi_k^+(x_0, x_1 - y_1)\epsilon^{k-1/2}\right) \\
\text{for } x_1 - y_1 > x_0, \\
\\
\exp\left(-\epsilon^{-1}\phi^-(x_0, x_1 - y_1)\right)\sum_{k=0}^{\infty}\psi_k^-(x_0, x_1 - y_1)\epsilon^{k-1/2} \\
\text{for } x_1 - y_1 < x_0,
\end{cases}$$

where

$$\begin{aligned}
\phi^+(x_0, x_1) &= (-2/(3\sqrt{3}))(x_1 - x_0)^{3/2}x_0^{-1/2} \\
\phi^-(x_0, x_1) &= (2/(3\sqrt{3}))(x_0 - x_1)^{3/2}x_0^{-1/2} \\
\psi_k^+(x_0, x_1) &= a_k(x_1 - x_0)^{-(3/2)k-1/4}x_0^{(1/2)k-1/4} \\
\psi_k^-(x_0, x_1) &= b_k(x_0 - x_1)^{-(3/2)k-1/4}x_0^{(1/2)k-1/4}
\end{aligned}$$

and where the a_k, b_k are given by (1.4), (1.5).
Thus:

$$\text{(2.17)} \qquad \operatorname{sing\,supp} H_\epsilon = \{(x_0, x_1) \in \overline{\mathbf{R}_+} \times \overline{\mathbf{R}_+} \mid x_1 \geq y_1 + x_0\}.$$

The operator G_ϵ is defined by:

$$\text{(2.18)} \quad (G_\epsilon f)(x_0, x_1) = \int_0^{x_0} \int_0^{\infty} H_\epsilon(x_0 - y_0, x_1, y_1) f(y_0, y_1) dy_1 dy_0.$$

Summarizing we have proved the following result:

Theorem 2.1 *The solution u_ϵ of* (2.1) – (2.4) *can be written in the form*

$$u_\epsilon = G_\epsilon f + H_\epsilon \phi + P_\epsilon^1 \psi_1 + P_\epsilon^2 \psi_2,$$

where the kernels of the operators H_ϵ, P_ϵ^1 and P_ϵ^2 are given by (2.14), (2.6) *and* (2.11) *respectively, their asymptotic expansions by* (2.16), (2.8) *and* (2.12) *respectively, and their singular supports are described by* (2.17), (2.10) *and* (2.13) *respectively. The operator G_ϵ is defined by* (2.18).

Remark. For singular perturbation (1.14) the mixed problem in the quarter-plane $\{x_0 > 0,\ x_1 > 0\}$ is not well-posed uniformly with respect to $\epsilon > 0$, since the corresponding characteristic equation

$$i\xi_0 + X + \epsilon^2 X^3 = 0,\ \xi_0 \in \mathbf{R},$$

does not have the same number of zeros generating stable solutions for $x_1 > 0$ for all $\epsilon \in (0, \epsilon_0), \forall \epsilon_0 < \infty$ ([4]).

Chapter 3

The mixed problem on a finite interval

Consider the following singularly perturbed initial-boundary value problem:

(3.1) $(\partial_{x_0} + \partial_{x_1} - \epsilon^2\partial_{x_1}^3)u(x_0, x_1) = f(x_0, x_1)$, $x_0 > 0$, $x_1 \in U = (0, 1)$,

(3.2) $u(0, x_1) = \phi(x_1)$, $\qquad x_1 \in U$,

(3.3) $u(x_0, 0) = \psi_1(x_0)$, $\qquad x_0 > 0$,

(3.4) $(\partial_{x_1} u)(x_0, 0) = \psi_2(x_0)$, $\qquad x_0 > 0$,

(3.5) $(\partial_{x_1} u)(x_0, 1) = \psi_3(x_0)$, $\qquad x_0 > 0$.

We write the solution u_ϵ of (3.1)-(3.5) in the following form:

$$u_\epsilon = G_\epsilon^U f + H_\epsilon^U \phi + P_\epsilon^{1,U}\psi_1 + P_\epsilon^{2,U}\psi_2 + P_\epsilon^{3,U}\psi_3,$$

where G_ϵ^U, H_ϵ^U, $P_\epsilon^{j,U}$ $(1 \leq j \leq 3)$ are integral operators. In this chapter we are going to determine the kernels of these operators, their asymptotic expansions as $\epsilon \downarrow 0$ and their singular supports.

First we consider the following problem:

(3.6) $\qquad (\partial_{x_0} + \partial_{x_1} - \epsilon^2\partial_{x_1}^3)u(x_0, x_1) = 0, \qquad x_0 > 0,\ x_1 < 1,$

(3.7) $\qquad u(0, x_1) = 0, \qquad x_1 < 1,$

(3.8) $\qquad (\partial_{x_1} u)(x_0, 1) = \delta(x_0), \qquad x_0 > 0.$

The solution of (3.6)-(3.8) is given by

(3.9) $P_\epsilon^3(x_0, x_1) =$

$$= (2\pi)^{-1}\int_{\mathbf{R}} (\mu_3(\xi_0))^{-1} \exp\left(\epsilon^{-1}\left(\mu_3(\xi_0)(x_1-1)+i\xi_0 x_0\right)\right) d\xi_0,$$

$x_0 > 0,\ x_1 < 1.$

Substituting $\eta = \mu_3(\xi_0)$ and using the fact that

$$(\partial_{\xi_0}\mu_3)(\xi_0) = i\left(3\left(\mu_3(\xi_0)\right)^2 - 1\right)^{-1},$$

one finds:

$$P_\epsilon^3(x_0, x_1) = (2\pi i)^{-1}\int_{\gamma_1} \eta^{-1}(3\eta^2-1)\exp\left(\epsilon^{-1}\left(\eta(x_1-1-x_0)+\eta^3 x_0\right)\right) d\eta,$$

where the curve γ_1 is given by the parametrization

$$\xi_0 \to \mu_3(\xi_0) \quad (\xi_0 \in \mathbf{R}).$$

For every $\xi_0 \in \mathbf{R}$ there is exactly one $\rho \in \mathbf{R}$ such that: $\rho^3 + \rho = \xi_0$. Thus the equation $i\xi_0 + \mu - \mu^3 = 0$ is solved by $\mu = -i\rho$ or $\mu = \pm(1/2)\cdot$ $\cdot(3\rho^2+4)^{1/2} + (1/2)i\rho$. Thus:

$$\mu_3(\xi_0) = (1/2)(3\rho^2+4)^{1/2} + (1/2)i\rho.$$

A new parametrization of the curve γ_1 is given by:

$$\gamma_1 : \tau \to (3\tau^2+1)^{1/2} + i\tau,\ \tau \in \mathbf{R}.$$

Define the curve γ_2 in the complex plane by the parametrization:

$$\gamma_2 : \tau \to \left((1/3)x_0^{-1}(x_0+1-x_1+\tau^2 x_0)\right)^{1/2} + i\tau,\ \tau \in \mathbf{R}.$$

For $|\tau|$ large enough we have:

$$\left|\int_{[\gamma_2(\tau),\gamma_1(\tau)]} \eta^{-1}(3\eta^2-1)\exp\left(\epsilon^{-1}\left(\eta(x_1-1-x_0)+\eta^3 x_0\right)\right) d\eta\right| \le$$

$$\le \int_{((1/3)x_0^{-1}(x_0+1-x_1+\tau^2 x_0))^{1/2}}^{(3\tau^2+1)^{1/2}} (t^2+\tau^2)^{-1/2}\left(\left(3(t^2-\tau^2)-1\right)^2 + 36t^2\tau^2\right)^{1/2}\cdot$$
$$\cdot\exp\left(\epsilon^{-1} t(x_1 - 1 - x_0 + t^2 x_0 - 3\tau^2 x_0)\right) dt \le$$

$$\leq 4|\tau|^{-1}(3\tau^2+1)\int_{\left((1/3)x_0^{-1}(x_0+1-x_1+\tau^2 x_0)\right)^{1/2}}^{(3\tau^2+1)^{1/2}} \exp\left(\epsilon^{-1}t(x_1-1)\right)dt =$$

$$= 4\epsilon|\tau|^{-1}(3\tau^2+1)(x_1-1)^{-1}\left(\exp\left(\epsilon^{-1}(x_1-1)(3\tau^2+1)^{1/2}\right)+\right.$$

$$\left. -\exp\left(\epsilon^{-1}(x_1-1)\left((1/3)x_0^{-1}(x_0+1-x_1+\tau^2 x_0)\right)^{1/2}\right)\right)$$

$$\to 0 \text{ as } |\tau| \to \infty.$$

Therefore we may rewrite (3.9) as follows:

$$P_\epsilon^3(x_0,x_1) = (2\pi i)^{-1}\int_{\gamma_2}\eta^{-1}(3\eta^2-1)\exp\left(\epsilon^{-1}\left(\eta(x_1-1-x_0)+\eta^3 x_0\right)\right)d\eta =$$

$$= (2\pi)^{-1}\int_{\mathbf{R}}\left((1/3)x_0^{-1}(x_0+1-x_1+\tau^2 x_0)\right)^{-1/2}\cdot$$

$$\cdot\left((1/3)x_0^{-1}(x_0+1-x_1+\tau^2 x_0)+\tau^2\right)^{-1}\cdot$$

$$\cdot\left((1/3)x_0^{-1}(1-x_1)+(1/3)x_0^{-2}(1-x_1)^2+\right.$$

$$\left.+2x_0^{-1}(1-x_1)\tau^2+(8/3)\tau^4+2\tau^2\right)\cdot$$

$$\cdot\exp\left(-\epsilon^{-1}\left((1/3)x_0^{-1}(x_0+1-x_1+\tau^2 x_0)\right)^{1/2}\cdot\right.$$

$$\left.\cdot\left((2/3)(x_0+1-x_1)+(8/3)x_0\tau^2\right)\right)d\tau,$$

where we again used the fact that the integral over $\mathbf{R}$ of an odd function is zero. Application of the saddle-point method to this last integral yields:

$$\text{(3.10)}\quad P_\epsilon^3(x_0,x_1) \sim \exp\left(\left(-2/(3\sqrt{3})\right)\epsilon^{-1}(x_0+1-x_1)^{3/2}x_0^{-1/2}\right)\cdot$$

$$\cdot\sum_{n=0}^{\infty}e_n(x_0,x_1)\epsilon^{n+1/2} \text{ as } \epsilon\downarrow 0,$$

where

$$e_0(x_0,x_1) = (1/2)\cdot 3^{1/4}\cdot\pi^{-1/2}\cdot(x_0+1-x_1)^{-3/4}\cdot x_0^{-3/4}\cdot(1-x_1).$$

Now define $P_\epsilon^{i,U}(x_0,x_1)$ as the solutions of (3.1)-(3.5) with $f=0, \phi=0$, $\psi_i(x_0)=\delta(x_0)$, $\psi_j=0$ if $j\neq i$, $1\leq i,j\leq 3$. Then we have:

$$\text{(3.11)}\quad P_\epsilon^{1,U}(x_0,x_1) = P_\epsilon^1(x_0,x_1)+$$

$$-\int_0^{x_0}(\partial_{x_1}P_\epsilon^1)(y_0,1)P_\epsilon^{3,U}(x_0-y_0,x_1)dy_0$$

$$(3.12)\quad P_\epsilon^{2,U}(x_0,x_1) = P_\epsilon^2(x_0,x_1)+ \\ -\int_0^{x_0}(\partial_{x_1}P_\epsilon^2)(y_0,1)P_\epsilon^{3,U}(x_0-y_0,x_1)dy_0$$

$$(3.13)\quad P_\epsilon^{3,U}(x_0,x_1) = P_\epsilon^3(x_0,x_1)+ \\ -\int_0^{x_0}P_\epsilon^3(y_0,0)P_\epsilon^{1,U}(x_0-y_0,x_1)dy_0+ \\ -\int_0^{x_0}(\partial_{x_1}P_\epsilon^3)(y_0,0)P_\epsilon^{2,U}(x_0-y_0,x_1)dy_0,$$

where P_ϵ^i $(1\le i\le 3)$ are given by (2.6), (2.11), (3.9). Substituting (3.11), (3.12) into (3.13) we get:

$$(3.14)\quad P_\epsilon^{3,U}(x_0,x_1) =$$

$$= P_\epsilon^3(x_0,x_1) - \int_0^{x_0}\Big(P_\epsilon^3(y_0,0)P_\epsilon^1(x_0-y_0,x_1)+(\partial_{x_1}P_\epsilon^3)(y_0,0)\cdot \\ \cdot P_\epsilon^2(x_0-y_0,x_1)\Big)\,dy_0+$$

$$+\int_0^{x_0}\int_0^{x_0-z_0}\Big(P_\epsilon^3(y_0,0)(\partial_{x_1}P_\epsilon^1)(x_0-y_0-z_0,1)+(\partial_{x_1}P_\epsilon^3)(y_0,0)\cdot \\ \cdot(\partial_{x_1}P_\epsilon^2)(x_0-y_0-z_0,1)\Big)\,dy_0P_\epsilon^{3,U}(z_0,x_1)dz_0 =$$

$$= K_\epsilon(x_0,x_1)+\int_0^{x_0}L_\epsilon(x_0-y_0)P_\epsilon^{3,U}(y_0,x_1)dy_0,$$

where

$$K_\epsilon(x_0,x_1) = P_\epsilon^3(x_0,x_1)+ \\ -\int_0^{x_0}\Big(P_\epsilon^3(y_0,0)P_\epsilon^1(x_0-y_0,x_1)+(\partial_{x_1}P_\epsilon^3)(y_0,0)P_\epsilon^2(x_0-y_0,x_1)\Big)\,dy_0$$

$$L_\epsilon(x_0) = \int_0^{x_0}\Big(P_\epsilon^3(y_0,0)(\partial_{x_1}P_\epsilon^1)(x_0-y_0,1)+ \\ +(\partial_{x_1}P_\epsilon^3)(y_0,0)(\partial_{x_1}P_\epsilon^2)(x_0-y_0,1)\Big)\,dy_0.$$

The Volterra integral equation (3.14) has the unique solution $P_\epsilon^{3,U}$. Furthermore $K_\epsilon(x_0,x_1)$ has the same asymptotic expansion as $P_\epsilon^3(x_0,x_1)$. Indeed using the asymptotic expansions (2.8), (2.12), (3.10) and a similar expansion for $\partial_{x_1}P_\epsilon^3$, this statement is a consequence of the following inequalities:

$$(x_0+1-x_1)^{3/2}x_0^{-1/2} < (y_0+1)^{3/2}y_0^{-1/2} \text{ for } x_0-x_1\le y_0\le x_0,\ y_0>0,$$

$$(x_0+1-x_1)^{3/2}x_0^{-1/2} < (y_0+1)^{3/2}y_0^{-1/2}+(x_0-y_0-x_1)^{3/2}(x_0-y_0)^{-1/2}$$
$$\text{for } 0<y_0<x_0-x_1<x_0,$$

where the first inequality is proved directly and the second inequality is exactly (2.15) with $\gamma = 1$, $\alpha = x_0 - x_1$, $\beta = y_0$, $t = x_0$.

The functions $u_\epsilon^n(x_0, x_1)$, $n = 1, 2, \ldots$, defined by

$$\begin{cases} u_\epsilon^1(x_0, x_1) = \displaystyle\int_0^{x_0} L_\epsilon(x_0 - y_0) K_\epsilon(y_0, x_1) dy_0 \\ \\ u_\epsilon^{n+1}(x_0, x_1) = \displaystyle\int_0^{x_0} L_\epsilon(x_0 - y_0) u_\epsilon^n(y_0, x_1) dy_0 \end{cases}$$

are asymptotically negligible compared to $K_\epsilon(x_0, x_1)$. Indeed, using the asymptotic expansions of $K_\epsilon, P_\epsilon^3, \partial_{x_1} P_\epsilon^i$ $(1 \le i \le 3)$, this statement is a consequence of the following inequalities:

$$(x_0 + 1 - x_1)^{3/2} x_0^{-1/2} < (y_0 + 1)^{3/2} y_0^{-1/2} + (z_0 + 1 - x_1)^{3/2} z_0^{-1/2}$$

$$\text{for } 0 < z_0 \le x_0,\ 0 < y_0 \le x_0 - z_0,\ x_0 - y_0 - z_0 \le 1,$$

$$(x_0 + 1 - x_1)^{3/2} x_0^{-1/2} < (y_0 + 1)^{3/2} y_0^{-1/2} +$$

$$+ (x_0 - y_0 - z_0 - 1)^{3/2} (x_0 - y_0 - z_0)^{-1/2} + (z_0 + 1 - x_1)^{3/2} z_0^{-1/2}$$

$$\text{for } 0 < z_0 \le x_0,\ 0 < y_0 \le x_0 - z_0,\ 1 < x_0 - y_0 - z_0.$$

The first inequality is proved in the following way: If $z_0 = x_0$ the inequality is immediate. If $z_0 < x_0$, then introduce the function

$$f(x_1) := (y_0+1)^{3/2} y_0^{-1/2} + (z_0+1-x_1)^{3/2} z_0^{-1/2} - (x_0+1-x_1)^{3/2} x_0^{-1/2} \ (x_1 < 1).$$

We have:

$$\begin{aligned} (\partial_{x_1} f)(x_1) &= -(3/2)(z_0 + 1 - x_1)^{1/2} z_0^{-1/2} + (3/2)(x_0 + 1 - x_1)^{1/2} x_0^{-1/2} = \\ &= -(3/2)(1 + (1 - x_1) z_0^{-1})^{1/2} + (3/2)(1 + (1 - x_1) x_0^{-1})^{1/2} < \\ &< 0 \text{ for } x_1 < 1, \end{aligned}$$

since $z_0 < x_0$. Therefore:

$$\begin{aligned} f(x_1) > \lim_{x_1 \uparrow 1} f(x_1) &= (y_0 + 1)^{3/2} y_0^{-1/2} + z_0 - x_0 = \\ &= ((y_0 + 1) y_0^{-1})^{1/2} (y_0 + 1) + z_0 - x_0 > \\ &> y_0 + 1 + z_0 - x_0 > 0. \end{aligned}$$

The second inequality is proved by using (2.15) twice: first with $\gamma = 1 - x_1$, $\beta = z_0$, $\alpha = x_0 - y_0 - 1$, $t = x_0 - y_0$, secondly with $\gamma = 1$, $\beta = y_0$, $\alpha =$

$x_0 - x_1$, $t = x_0$. Since $P_\epsilon^{3,U}(x_0,x_1) = K_\epsilon(x_0,x_1) + \sum_{n=1}^{\infty} u_\epsilon^n(x_0,x_1)$, we reach the conclusion:
$P_\epsilon^{3,U}(x_0,x_1)$ has the same asymptotic expansion as $K_\epsilon(x_0,x_1)$ as $\epsilon \downarrow 0$, the latter being given by (3.10). Moreover the singular support of $P_\epsilon^{3,U}$ consists of just one point:

$$\text{sing supp } P_\epsilon^{3,U} = \{(0,1)\}. \tag{3.15}$$

$P_\epsilon^{i,U}$ $(i = 1,2)$ are given by (3.11), (3.12). Using the asymptotic expansions of $P_\epsilon^i, \partial_{x_1} P_\epsilon^i$ $(i = 1,2)$ and $P_\epsilon^{3,U}$ and the following inequalities:

$$(x_0 - x_1)^{3/2} x_0^{-1/2} < (x_0 - y_0 + 1 - x_1)^{3/2}(x_0 - y_0)^{-1/2}$$
$$\text{for } x_0 > x_1,\ 0 < y_0 < x_0,\ y_0 \le 1,$$
$$(x_0 - x_1)^{3/2} x_0^{-1/2} < (y_0 - 1)^{3/2} y_0^{-1/2} + (x_0 - y_0 + 1 - x_1)^{3/2}(x_0 - y_0)^{-1/2}$$
$$\text{for } x_0 > x_1,\ 0 < y_0 < x_0,\ y_0 > 1,$$

(the first being proved directly, the second being inequality (2.15) with $\gamma = 1 - x_1$, $\beta = x_0 - y_0$, $\alpha = x_0 - 1$, $t = x_0$), one comes to the conclusion: $P_\epsilon^{i,U}$ has the same asymptotic expansion as P_ϵ^i for $\epsilon \downarrow 0$ $(i = 1,2)$, given by (2.8), (2.12).
Thus:

$$\text{sing supp } P_\epsilon^{1,U} = \text{sing supp } P_\epsilon^{2,U} = \{(x_0,x_1) \in \overline{\mathbf{R}_+} \times \overline{U} \mid x_0 \le x_1\}. \tag{3.16}$$

The integral operators $P_\epsilon^{i,U}, 1 \le i \le 3$ are defined in the following way:

$$(P_\epsilon^{i,U}\psi_i)(x_0,x_1) = \int_0^{x_0} P_\epsilon^{i,U}(x_0 - y_0, x_1)\psi_i(y_0)dy_0. \tag{3.17}$$

The solution $H_\epsilon^U(x_0,x_1,y_1)$ of (3.1)-(3.5) with $f = 0$, $\phi = \delta(x_1 - y_1)$, $\psi_i = 0$, $1 \le i \le 3$ may be written in the following form

$$\begin{aligned} H_\epsilon^U(x_0,x_1,y_1) = \; & E_\epsilon(x_0, x_1 - y_1) + \\ & -\int_0^{x_0} E_\epsilon(y_0, -y_1) P_\epsilon^{1,U}(x_0 - y_0, x_1)dy_0 + \\ & -\int_0^{x_0} (\partial_{x_1} E_\epsilon)(y_0, -y_1) P_\epsilon^{2,U}(x_0 - y_0, x_1)dy_0 + \\ & -\int_0^{x_0} (\partial_{x_1} E_\epsilon)(y_0, 1 - y_1) P_\epsilon^{3,U}(x_0 - y_0, x_1)dy_0, \end{aligned} \tag{3.18}$$

where E_ϵ is defined by (1.3). Using the asymptotic expansions of $E_\epsilon, \partial_{x_1} E_\epsilon$ and $P_\epsilon^{i,U}$ $(i = 1, 2, 3)$ and the following inequalities:

$$(x_0 - x_1 + y_1)^{3/2} x_0^{-1/2} < (y_0 + y_1)^{3/2} y_0^{-1/2}$$
$$\text{for } x_0 > x_1 - y_1,\ x_0 - x_1 \leq y_0 \leq x_0,\ y_0 > 0,$$

$$(x_0 - x_1 + y_1)^{3/2} x_0^{-1/2} < (y_0 + y_1)^{3/2} y_0^{-1/2} + (x_0 - y_0 - x_1)^{3/2} (x_0 - y_0)^{-1/2}$$
$$\text{for } x_0 > x_1 - y_1,\ 0 < y_0 < x_0 - x_1,$$

$$(x_0 - x_1 + y_1)^{3/2} x_0^{-1/2} < (x_0 - y_0 + 1 - x_1)^{3/2} (x_0 - y_0)^{-1/2}$$
$$\text{for } x_0 > x_1 - y_1,\ 0 < y_0 < x_0,\ y_0 \leq 1 - y_1,$$

$$(x_0 - x_1 + y_1)^{3/2} x_0^{-1/2} < (y_0 - 1 + y_1)^{3/2} y_0^{-1/2} + $$
$$+ (x_0 - y_0 + 1 - x_1)^{3/2} (x_0 - y_0)^{-1/2}$$
$$\text{for } x_0 > x_1 - y_1,\ 0 < y_0 < x_0,\ y_0 > 1 - y_1,$$

one finds that $H_\epsilon^U(x_0, x_1, y_1)$ and $E_\epsilon(x_0, x_1 - y_1)$ have the same asymptotic expansions as $\epsilon \downarrow 0$. The first and second inequalities have been proved in chapter 2. The third inequality is proved straightforward and the fourth inequality is (2.15) with $\gamma = 1 - x_1$, $\beta = x_0 - y_0$, $\alpha = x_0 + y_1 - 1$, $t = x_0$. The singular support of $H_\epsilon^U(x_0, x_1, y_1)$ is given by:

$$\text{(3.19)} \qquad \text{sing supp } H_\epsilon^U = \{(x_0, x_1) \in \overline{\mathbf{R}_+} \times \overline{U} \mid x_0 \leq x_1 - y_1\}$$

and the corresponding integral operator H_ϵ^U is defined by

$$\text{(3.20)} \qquad (H_\epsilon^U \phi)(x_0, x_1) = \int_U H_\epsilon^U(x_0, x_1, y_1) \phi(y_1) dy_1.$$

The integral operator G_ϵ^U is defined by

$$\text{(3.21)} \quad (G_\epsilon^U f)(x_0, x_1) = \int_0^{x_0} \int_U H_\epsilon^U(x_0 - y_0, x_1, y_1) f(y_0, y_1) dy_1 dy_0.$$

Summarizing we have proved the following result:

Theorem 3.1 *The solution u_ϵ of (3.1) – (3.5) can be written in the form*

$$u_\epsilon = G_\epsilon^U f + H_\epsilon^U \phi + P_\epsilon^{1,U} \psi_1 + P_\epsilon^{2,U} \psi_2 + P_\epsilon^{3,U} \psi_3,$$

where the operators $G_\epsilon^U, H_\epsilon^U, P_\epsilon^{j,U}$ $(1 \leq j \leq 3)$ are respectively defined by (3.21), (3.20) and (3.17). Kernel $P_\epsilon^{3,U}(x_0, x_1)$ is the unique solution of Volterra-integral equation (3.14) and its asymptotic expansion is given by (3.10). The *kernels* $P_\epsilon^{j,U}$ $(j = 1, 2)$, *solutions of* (3.11), (3.12), *have their asymptotic expansions given by* (2.8), (2.12) *and kernel* $H_\epsilon^U(x_0, x_1, y_1)$ *is given by* (3.18), *its asymptotic expansion is given by* (2.16). *The singular supports of the kernels* $P_\epsilon^{j,U}(x_0, x_1)$ $(1 \le j \le 3)$ *and* $H_\epsilon^U(x_0, x_1, y_1)$ *are respectively described by* (3.16), (3.15) *and* (3.19).

Chapter 4

The eigenvalue problem

Here the eigenvalue problem for the stationary part of the operator (3.1)-(3.5) is considered, where, for the sake of convenience, a normalizing factor 1/3 is introduced. A different approach based on the general theory of coercive singular perturbations, has been used in [5] for the asymptotic analysis of this eigenvalue problem.

Thus consider:

$$(\partial_x - (1/3)\epsilon^2\partial_x^3)u(x) = \lambda u(x),\ x \in U \tag{4.1}$$

$$u(0) = (\partial_x u)(0) = (\partial_x u)(1) = 0. \tag{4.2}$$

Eigenvalue problem (4.1), (4.2) is equivalent to the following singularly perturbed eigenvalue problem:

$$(\epsilon\partial_x - (1/3)\epsilon^3\partial_x^3)u(x) = \mu u(x),\ x \in U \tag{4.3}$$

$$u(0) = (\partial_x u)(0) = (\partial_x u)(1) = 0, \tag{4.2}$$

i.e. λ is an eigenvalue of (4.1), (4.2) associated with eigenfunction u iff $\mu = \epsilon\lambda$ is an eigenvalue of (4.3), (4.2) associated with the same eigenfunction. In this chapter we are going to determine for all eigenvalues and eigenfunctions of (4.3), (4.2) asymptotic formulae as $\epsilon \downarrow 0$.

The solution of (4.3), satisfying the initial conditions $u(0) = 0$, $(\partial_x u)(0) = 0$, $(\partial_x^2 u)(0) = -\epsilon^{-2}$, is given by the following expression:

$$\begin{aligned} u(x) = \ & [(\lambda_3(\mu) - \lambda_2(\mu))\exp(\epsilon^{-1}\lambda_1(\mu)x) + \\ & +(\lambda_1(\mu) - \lambda_3(\mu))\exp(\epsilon^{-1}\lambda_2(\mu)x) + \\ & +(\lambda_2(\mu) - \lambda_1(\mu))\exp(\epsilon^{-1}\lambda_3(\mu)x)] \cdot \\ & \cdot[(\lambda_3(\mu) - \lambda_2(\mu))(\lambda_1(\mu) - \lambda_3(\mu))(\lambda_2(\mu) - \lambda_1(\mu))]^{-1}, \end{aligned} \tag{4.4}$$

where the $\lambda_i(\mu)\,(1 \le i \le 3)$ are the three zeros of the equation:

$$X - (1/3)X^3 - \mu = 0. \tag{4.5}$$

For every $x \in \overline{U}$, the right-hand side of (4.4) is an entire function of μ. (For $\mu = \pm 2/3$ equation (4.5) has multiple zeros and the right-hand side of (4.4) is defined by analytic continuation.)

Differentiation of (4.4) yields:

$$\begin{aligned}(\partial_x u)(x) = \epsilon^{-1}\ & [\lambda_1(\mu)(\lambda_3(\mu) - \lambda_2(\mu))\exp(\epsilon^{-1}\lambda_1(\mu)x) + \\ & +\lambda_2(\mu)(\lambda_1(\mu) - \lambda_3(\mu))\exp(\epsilon^{-1}\lambda_2(\mu)x) + \\ & +\lambda_3(\mu)(\lambda_2(\mu) - \lambda_1(\mu))\exp(\epsilon^{-1}\lambda_3(\mu)x)]\cdot \\ & \cdot[(\lambda_3(\mu) - \lambda_2(\mu))(\lambda_1(\mu) - \lambda_3(\mu))(\lambda_2(\mu) - \lambda_1(\mu))]^{-1},\end{aligned}$$

which is again, for every $x \in \overline{U}$ an entire function of μ. Substituting $x = 1$, we reach the conclusion:

μ is an eigenvalue of (4.3), (4.2)

iff

$$\begin{aligned}D(\epsilon,\mu) :=\ & [\lambda_1(\mu)(\lambda_3(\mu) - \lambda_2(\mu))\exp(\epsilon^{-1}\lambda_1(\mu)) + \\ & +\lambda_2(\mu)(\lambda_1(\mu) - \lambda_3(\mu))\exp(\epsilon^{-1}\lambda_2(\mu)) + \\ & +\lambda_3(\mu)(\lambda_2(\mu) - \lambda_1(\mu))\exp(\epsilon^{-1}\lambda_3(\mu))]\cdot \\ & \cdot[(\lambda_3(\mu) - \lambda_2(\mu))(\lambda_1(\mu) - \lambda_3(\mu))(\lambda_2(\mu) - \lambda_1(\mu))]^{-1} = 0,\end{aligned}$$

where $D(\epsilon,\mu)$ is an entire function of μ for every $\epsilon > 0$.

Lemma 4.1 *If $\mu \in \mathbf{R}$, $\mu \le 2/3$ then $D(\epsilon,\mu) \ne 0$ for every $\epsilon > 0$.*

Proof. Let $\epsilon > 0$. First we show that $D(\epsilon,\mu)$ has no strictly negative zeros $\mu(\epsilon)$. For $\mu(\epsilon)$ a real zero of $D(\epsilon,\mu)$ exists a real-valued function u, solution of the eigenvalue problem:

$$\begin{cases} (\epsilon\partial_x - (1/3)\epsilon^3\partial_x^3)u(x) = \mu(\epsilon)u(x),\ x \in U \\ u(0) = (\partial_x u)(0) = (\partial_x u)(1) = 0, \end{cases}$$

satisfying the condition $(\partial_x^2 u)(0) = 1$. Since $(\partial_x u)(x') = 0$, $x' \in \partial U = \{0,1\}$, the mean-value theorem yields for some $a \in U$:

$$\begin{cases} (\partial_x^2 u)(a) = 0 \\ (\partial_x^2 u)(y) > 0 \text{ for every } y \in [0,a). \end{cases}$$

Therefore: $u(y) > 0 \ \forall y \in (0, a)$.
Thus one finds:

$$\begin{aligned}
\mu(\epsilon)\int_0^a (u(x))^2 dx &= \int_0^a \left(\epsilon(\partial_x u)(x) - (1/3)\epsilon^3(\partial_x^3 u)(x)\right) u(x)dx = \\
&= (1/2)\epsilon\,(u(a))^2 + (1/3)\epsilon^3\int_0^a (\partial_x^2 u)(x)(\partial_x u)(x)dx = \\
&= (1/2)\epsilon\,(u(a))^2 + (1/6)\epsilon^3\left((\partial_x u)(a)\right)^2 \geq 0,
\end{aligned}$$

so that for each real zero $\mu(\epsilon)$ of $D(\epsilon, \mu)$ one has: $\mu(\epsilon) \geq 0$.
If $\mu \in [0, 2/3)$ the zeros $\lambda_i(\mu)$ $(1 \leq i \leq 3)$ are real and may be ordered as follows:

$$\lambda_1(\mu) \leq -3^{1/2} < 0 \leq \lambda_2(\mu) < 1 < \lambda_3(\mu) \leq 3^{1/2},$$

so that

$$\begin{aligned}
D(\epsilon,\mu) = \ & [(\lambda_3(\mu) - \lambda_2(\mu))\,(\lambda_1(\mu) - \lambda_3(\mu))\,(\lambda_2(\mu) - \lambda_1(\mu))]^{-1}\cdot \\
& \cdot[\lambda_2(\mu)\,(\lambda_3(\mu) - \lambda_1(\mu))\,(\exp(\epsilon^{-1}\lambda_3(\mu)) - \exp(\epsilon^{-1}\lambda_2(\mu))) + \\
& + \lambda_1(\mu)\,(\lambda_3(\mu) - \lambda_2(\mu))\,(\exp(\epsilon^{-1}\lambda_1(\mu)) - \exp(\epsilon^{-1}\lambda_3(\mu)))] < 0.
\end{aligned}$$

If $\mu = 2/3$, then

$$\begin{aligned}
D(\epsilon, 2/3) &= -(1/3)\epsilon^{-1}\exp(\epsilon^{-1}) + (2/9)\exp(-2\epsilon^{-1}) - (2/9)\exp(\epsilon^{-1}) = \\
&= -(1/3)\epsilon^{-1}\exp(\epsilon^{-1})\,(1 + (2/3)\epsilon - (2/3)\epsilon\exp(-3\epsilon^{-1})) \neq 0.
\end{aligned}$$

□

In the next part we will prove that all zeros of $D(\epsilon, \mu)$ are real and positive for ϵ sufficiently small. In order to do this we need some information about the zeros $\lambda_i(\mu)$ $(1 \leq i \leq 3)$ of equation (4.5).

The region $V := \mathbf{C}\backslash(-\infty, 2/3]$ is simply connected and for every $\mu \in V$ equation (4.5) has no multiple zeros. Therefore the zeros $\lambda_i(\mu)$ $(1 \leq i \leq 3)$ may be defined as holomorphic functions in V. We do that in such a way that

$$\begin{cases}
\lambda_1\left((4/3)i\right) = -(1/2)\sqrt{15} - (1/2)i \\
\lambda_2\left((4/3)i\right) = i \\
\lambda_3\left((4/3)i\right) = (1/2)\sqrt{15} - (1/2)i.
\end{cases}$$

Then one finds the following relations for $\eta \in \{z \in \mathbf{C} \,|\, |z - (4/3)i| < 4/3\}$:

$$\begin{cases}
\lambda_1(\eta) = 3^{1/2}\mu_2(3^{-1/2}i\eta) \\
\lambda_2(\eta) = 3^{1/2}\mu_1(3^{-1/2}i\eta) \\
\lambda_3(\eta) = 3^{1/2}\mu_3(3^{-1/2}i\eta),
\end{cases}$$

where μ_j $(1 \le j \le 3)$ were defined in chapter 2. One easily verifies the following inequalities in different parts of V:

$$\begin{cases} \text{Re } \lambda_1(\mu) < \text{Re } \lambda_2(\mu) < 0 < \text{Re } \lambda_3(\mu) \\ \text{Im } \lambda_1(\mu) < \text{Im } \lambda_3(\mu) < 0 < \text{Im } \lambda_2(\mu) \end{cases} \qquad \text{if Re } \mu < 0, \text{ Im } \mu > 0,$$

$$\begin{cases} \text{Re } \lambda_1(\mu) < 0 = \text{Re } \lambda_2(\mu) < \text{Re } \lambda_3(\mu) \\ \text{Im } \lambda_1(\mu) = \text{Im } \lambda_3(\mu) < 0 < \text{Im } \lambda_2(\mu) \end{cases} \qquad \text{if Re } \mu = 0, \text{ Im } \mu > 0,$$

$$\begin{cases} \text{Re } \lambda_1(\mu) < 0 < \text{Re } \lambda_2(\mu) < \text{Re } \lambda_3(\mu) \\ \text{Im } \lambda_3(\mu) < \text{Im } \lambda_1(\mu) < 0 < \text{Im } \lambda_2(\mu) \end{cases} \qquad \text{if Re } \mu > 0, \text{ Im } \mu > 0,$$

$$\begin{cases} \text{Re } \lambda_1(\mu) < 0 < \text{Re } \lambda_2(\mu) = \text{Re } \lambda_3(\mu) \\ \text{Im } \lambda_3(\mu) < 0 = \text{Im } \lambda_1(\mu) < \text{Im } \lambda_2(\mu) \end{cases} \qquad \text{if Re } \mu > 2/3, \text{ Im } \mu = 0,$$

$$\begin{cases} \text{Re } \lambda_1(\mu) < 0 < \text{Re } \lambda_3(\mu) < \text{Re } \lambda_2(\mu) \\ \text{Im } \lambda_3(\mu) < 0 < \text{Im } \lambda_1(\mu) < \text{Im } \lambda_2(\mu) \end{cases} \qquad \text{if Re } \mu > 0, \text{ Im } \mu < 0,$$

$$\begin{cases} \text{Re } \lambda_1(\mu) < 0 = \text{Re } \lambda_3(\mu) < \text{Re } \lambda_2(\mu) \\ \text{Im } \lambda_3(\mu) < 0 < \text{Im } \lambda_1(\mu) = \text{Im } \lambda_2(\mu) \end{cases} \qquad \text{if Re } \mu = 0, \text{ Im } \mu < 0,$$

$$\begin{cases} \text{Re } \lambda_1(\mu) < \text{Re } \lambda_3(\mu) < 0 < \text{Re } \lambda_2(\mu) \\ \text{Im } \lambda_3(\mu) < 0 < \text{Im } \lambda_2(\mu) < \text{Im } \lambda_1(\mu) \end{cases} \qquad \text{if Re } \mu < 0, \text{ Im } \mu < 0,$$

Let $0 < \epsilon \le 1/2$, $\mu \in V$ be such that $\text{Re}\,(\lambda_3(\mu) - \lambda_2(\mu)) = \epsilon$. Writing $\lambda_3(\mu) - \lambda_2(\mu) = \epsilon - it$ where $t > 0$ and setting $p = \lambda_1(\mu)$ we have:

$$\begin{cases} \lambda_2(\mu) = -(p/2) + (1/2)(12 - 3p^2)^{1/2} \\ \lambda_3(\mu) = -(p/2) - (1/2)(12 - 3p^2)^{1/2}, \end{cases}$$

where $(12 - 3p^2)^{1/2}$ is chosen with $\text{Im}\left((12 - 3p^2)^{1/2}\right) > 0$.
Therefore: $-(12 - 3p^2)^{1/2} = \epsilon - it$.
One has:

$$p^2 = (1/3)(12 + t^2 - \epsilon^2) + (2/3)i\epsilon t,$$

$$\begin{aligned} p = \; & -\left\{(1/6)(12 + t^2 - \epsilon^2) + [((1/6)(12 + t^2 - \epsilon^2))^2 + ((1/3)t\epsilon)^2]^{1/2}\right\}^{1/2} + \\ & -i\left\{-(1/6)(12 + t^2 - \epsilon^2) + [((1/6)(12 + t^2 - \epsilon^2))^2 + ((1/3)t\epsilon)^2]^{1/2}\right\}^{1/2}, \end{aligned}$$

so that

$$(4.6)\qquad \mu(t,\epsilon) = p - (1/3)p^3 =$$

$$= (1/(9\sqrt{6}))\left\{12 + t^2 - \epsilon^2 + \left((12 + t^2 - \epsilon^2)^2 + 4t^2\epsilon^2\right)^{1/2}\right\}^{1/2} \cdot$$

$$\cdot\left\{-9 + 2(12 + t^2 - \epsilon^2) - \left((12 + t^2 - \epsilon^2)^2 + 4t^2\epsilon^2\right)^{1/2}\right\} +$$

$$+i(1/(9\sqrt{6}))\left\{-(12 + t^2 - \epsilon^2) + \left((12 + t^2 - \epsilon^2)^2 + 4t^2\epsilon^2\right)^{1/2}\right\}^{1/2} \cdot$$

$$\cdot\left\{-9 + 2(12 + t^2 - \epsilon^2) + \left((12 + t^2 - \epsilon^2)^2 + 4t^2\epsilon^2\right)^{1/2}\right\}.$$

On the other hand $\mu(t,\epsilon)$, given by (4.6), satisfies the relation:

$$\mathrm{Re}\left(\lambda_3\left(\mu(t,\epsilon)\right) - \lambda_2\left(\mu(t,\epsilon)\right)\right) = \epsilon \ \forall t > 0.$$

Therefore:

$$\begin{aligned}\Gamma_\epsilon : &= \{\mu \in V \mid \mathrm{Re}\left(\lambda_3(\mu) - \lambda_2(\mu)\right) = \epsilon\}\\ &= \{\mu(t,\epsilon) \mid t > 0\}.\end{aligned}$$

We also have:

$$\Gamma_{-\epsilon} := \{\mu \in V \mid \mathrm{Re}\left(\lambda_3(\mu) - \lambda_2(\mu)\right) = -\epsilon\} = \{\overline{\mu} \mid \mu \in \Gamma_\epsilon\}.$$

For $\epsilon \in (0,1/2]$ the two curves $\Gamma_{-\epsilon}$ and Γ_ϵ divide V into two regions U_ϵ and W_ϵ, where

$$U_\epsilon := \{\mu \in V \mid |\mathrm{Re}\left(\lambda_3(\mu) - \lambda_2(\mu)\right)| > \epsilon\}$$

$$W_\epsilon := \{\mu \in V \mid |\mathrm{Re}\left(\lambda_3(\mu) - \lambda_2(\mu)\right)| < \epsilon\}.$$

Note that $W_\epsilon \subset \{\mu \in V \mid \mathrm{Re}\,\mu > 0\}$ for $\epsilon \in (0,1/2]$.
We need the following lemma:

Lemma 4.2 *For ϵ sufficiently small the following inclusion holds:*

$$\{\mu \in V \mid D(\epsilon,\mu) = 0\} \subset W_\epsilon.$$

Proof. First, we prove the existence of a constant $\epsilon_1 > 0$ such that

$$\{\mu \in V \mid D(\epsilon,\mu) = 0\} \subset \{\mu \in V \mid |\mu + 2/3| \geq 1/10\} \text{ for every } \epsilon \in (0,\epsilon_1].$$

We have:

$$(4.7)\qquad D(\epsilon,\mu) = \left(\lambda_3(\mu) - \lambda_1(\mu)\right)^{-1}\exp(\epsilon^{-1}\lambda_1(\mu)) +$$

$$
\begin{aligned}
&+\epsilon^{-1}\lambda_2(\mu)\,(\lambda_3(\mu)-\lambda_1(\mu))^{-1}\,(\exp(\epsilon^{-1}\lambda_1(\mu))-\exp(\epsilon^{-1}\lambda_2(\mu)))\cdot\\
&\qquad\cdot(\epsilon^{-1}\,(\lambda_1(\mu)-\lambda_2(\mu)))^{-1}+\\
&+\lambda_2(\mu)\,((\lambda_2(\mu)-\lambda_3(\mu))\,(\lambda_1(\mu)-\lambda_3(\mu)))^{-1}\exp(\epsilon^{-1}\lambda_2(\mu))+\\
&+\lambda_3(\mu)\,((\lambda_1(\mu)-\lambda_3(\mu))\,(\lambda_3(\mu)-\lambda_2(\mu)))^{-1}\exp(\epsilon^{-1}\lambda_3(\mu)).
\end{aligned}
$$

The first three terms on the right-hand side of (4.7) are exponentially small as $\epsilon\downarrow 0$, uniformly with respect to $\mu\in\{z\in V\,|\,|z+2/3|<1/10,\,\mathrm{Im}\,z>0\}$; the fourth one is, however, exponentially large. Therefore using the relation $D(\epsilon,\overline{\mu})=\overline{D(\epsilon,\mu)}$, we find for ϵ sufficiently small:

$$
\{\mu\in V\,|\,D(\epsilon,\mu)=0\}\subset\{\mu\in V\,|\,|\mu+2/3|\geq 1/10\}\,.
$$

The functions:

$$
\begin{aligned}
&\mu\to\lambda_1(\mu)\,(\lambda_3(\mu))^{-1}\,(\lambda_3(\mu)-\lambda_2(\mu))\,(\lambda_2(\mu)-\lambda_1(\mu))^{-1}\\
&\mu\to\lambda_2(\mu)\,(\lambda_3(\mu))^{-1}\,(\lambda_1(\mu)-\lambda_3(\mu))\,(\lambda_2(\mu)-\lambda_1(\mu))^{-1}
\end{aligned}
$$

are bounded in the domain $\{\mu\in V\,|\,\mathrm{Im}\;\mu>0,|\mu+2/3|\geq 1/10\}$ by a constant $K>0$.
For $\mu\in V$ satisfying $\mathrm{Im}\;\mu>0$, $|\mu+2/3|\geq 1/10$, $D(\epsilon,\mu)=0$, one has:

$$
\begin{aligned}
&\lambda_1(\mu)\,(\lambda_3(\mu))^{-1}\,(\lambda_3(\mu)-\lambda_2(\mu))\,(\lambda_2(\mu)-\lambda_1(\mu))^{-1}\cdot\\
&\qquad\cdot\exp\left(\epsilon^{-1}\,(\lambda_1(\mu)-\lambda_3(\mu))\right)+\\
&+\lambda_2(\mu)\,(\lambda_3(\mu))^{-1}\,(\lambda_1(\mu)-\lambda_3(\mu))\,(\lambda_2(\mu)-\lambda_1(\mu))^{-1}\cdot\\
&\qquad\cdot\exp\left(\epsilon^{-1}\,(\lambda_2(\mu)-\lambda_3(\mu))\right)+\\
&+1=0.
\end{aligned}
$$

Thus:

$$
\begin{aligned}
1\;=\;&|\lambda_1(\mu)\,(\lambda_3(\mu))^{-1}\,(\lambda_3(\mu)-\lambda_2(\mu))\,(\lambda_2(\mu)-\lambda_1(\mu))^{-1}\cdot\\
&\qquad\cdot\exp\left(\epsilon^{-1}\,(\lambda_1(\mu)-\lambda_3(\mu))\right)+\\
&+\lambda_2(\mu)\,(\lambda_3(\mu))^{-1}\,(\lambda_1(\mu)-\lambda_3(\mu))\,(\lambda_2(\mu)-\lambda_1(\mu))^{-1}\cdot\\
&\qquad\cdot\exp\left(\epsilon^{-1}\,(\lambda_2(\mu)-\lambda_3(\mu))\right)|\leq\\
&\leq K\left\{\exp\left(\epsilon^{-1}\mathrm{Re}\,(\lambda_1(\mu)-\lambda_3(\mu))\right)+\exp\left(\epsilon^{-1}\mathrm{Re}\,(\lambda_2(\mu)-\lambda_3(\mu))\right)\right\}\leq\\
&\leq 2K\exp\left(\epsilon^{-1}\mathrm{Re}\,(\lambda_2(\mu)-\lambda_3(\mu))\right).
\end{aligned}
$$

Therefore one has:

$$0 \leq \mathrm{Re}\,(\lambda_3(\mu) - \lambda_2(\mu)) \leq \epsilon \log(2K).$$

Again, since $D(\epsilon,\overline{\mu}) = \overline{D(\epsilon,\mu)}$, we find for ϵ sufficiently small:

$$\{\mu \in V \mid D(\epsilon,\mu) = 0\} \subset \left\{\mu \in V \mid \epsilon^{-1}|\mathrm{Re}\,(\lambda_3(\mu) - \lambda_2(\mu))| \leq \log(2K)\right\}.$$

As a consequence, we have for ϵ sufficiently small:

$$\{\mu \in V \mid D(\epsilon,\mu) = 0\} \subset \{\mu \in V \mid \mathrm{Re}\,\mu > 0\}.$$

But in $\{\mu \in V \mid \mathrm{Re}\,\mu > 0, \mathrm{Im}\,\mu > 0\}$ the functions

$$\mu \to \lambda_1(\mu)\,(\lambda_3(\mu))^{-1}\,(\lambda_3(\mu) - \lambda_2(\mu))\,(\lambda_2(\mu) - \lambda_1(\mu))^{-1}$$
$$\mu \to \lambda_2(\mu)\,(\lambda_3(\mu))^{-1}\,(\lambda_1(\mu) - \lambda_3(\mu))\,(\lambda_2(\mu) - \lambda_1(\mu))^{-1}$$

are bounded by the constant $K = 1$. Therefore we have for ϵ sufficiently small:

$$\{\mu \in V \mid D(\epsilon,\mu) = 0\} \subset \left\{\mu \in V \mid \epsilon^{-1}|\mathrm{Re}\,(\lambda_3(\mu) - \lambda_2(\mu))| \leq \log 2\right\} \subset W_\epsilon.$$

□

Lemma 4.3 *There is an* $\epsilon_0 > 0$ *such that* $\epsilon \in (0,\epsilon_0]$, $\mu \in W_\epsilon$, $D(\epsilon,\mu) = 0$ *implies:* $(\partial_\mu D)(\epsilon,\mu) \neq 0$.

Proof. $D(\epsilon,\mu) = P(\mu)R(\epsilon,\mu)$, where

$$\begin{aligned}
P(\mu) &= [(\lambda_3(\mu) - \lambda_2(\mu))\,(\lambda_1(\mu) - \lambda_3(\mu))\,(\lambda_2(\mu) - \lambda_1(\mu))]^{-1} \\
R(\epsilon,\mu) &= \lambda_1(\mu)\,(\lambda_3(\mu) - \lambda_2(\mu))\exp(\epsilon^{-1}\lambda_1(\mu)) + \\
&\quad + \lambda_2(\mu)\,(\lambda_1(\mu) - \lambda_3(\mu))\exp(\epsilon^{-1}\lambda_2(\mu)) + \\
&\quad + \lambda_3(\mu)\,(\lambda_2(\mu) - \lambda_1(\mu))\exp(\epsilon^{-1}\lambda_3(\mu)).
\end{aligned}$$

We have:

$$(\partial_\mu D)(\epsilon,\mu) = (\partial_\mu P)(\mu)R(\epsilon,\mu) + P(\mu)(\partial_\mu R)(\epsilon,\mu).$$

Let $\mu \in W_\epsilon, D(\epsilon,\mu) = 0$, then $R(\epsilon,\mu) = 0$. Therefore

$$(\partial_\mu D)(\epsilon,\mu) = P(\mu)(\partial_\mu R)(\epsilon,\mu),$$

where:

$$
\begin{aligned}
&(\partial_\mu R)(\epsilon,\mu) = \\
&= \Big\{(\lambda_3(\mu)-\lambda_2(\mu))\left(1-(\lambda_1(\mu))^2\right)^{-1} + \\
&\quad +\lambda_1(\mu)\left(1-(\lambda_3(\mu))^2\right)^{-1} - \lambda_1(\mu)\left(1-(\lambda_2(\mu))^2\right)^{-1} + \\
&\quad +\epsilon^{-1}\lambda_1(\mu)(\lambda_3(\mu)-\lambda_2(\mu))\left(1-(\lambda_1(\mu))^2\right)^{-1}\Big\}\exp(\epsilon^{-1}\lambda_1(\mu)) + \\
&\quad +\Big\{(\lambda_1(\mu)-\lambda_3(\mu))\left(1-(\lambda_2(\mu))^2\right)^{-1} + \\
&\quad +\lambda_2(\mu)\left(1-(\lambda_1(\mu))^2\right)^{-1} - \lambda_2(\mu)\left(1-(\lambda_3(\mu))^2\right)^{-1} + \\
&\quad +\epsilon^{-1}\lambda_2(\mu)(\lambda_1(\mu)-\lambda_3(\mu))\left(1-(\lambda_2(\mu))^2\right)^{-1}\Big\}\exp(\epsilon^{-1}\lambda_2(\mu)) + \\
&\quad +\Big\{(\lambda_2(\mu)-\lambda_1(\mu))\left(1-(\lambda_3(\mu))^2\right)^{-1} + \\
&\quad +\lambda_3(\mu)\left(1-(\lambda_2(\mu))^2\right)^{-1} - \lambda_3(\mu)\left(1-(\lambda_1(\mu))^2\right)^{-1} + \\
&\quad +\epsilon^{-1}\lambda_3(\mu)(\lambda_2(\mu)-\lambda_1(\mu))\left(1-(\lambda_3(\mu))^2\right)^{-1}\Big\}\exp(\epsilon^{-1}\lambda_3(\mu)).
\end{aligned}
$$

Here we used that $(\partial_\mu\lambda_i)(\mu) = \left(1-(\lambda_i(\mu))^2\right)^{-1}$ $(1 \le i \le 3)$.
Since $R(\epsilon,\mu) = 0$ in the domain considered, we find:

$$
\begin{aligned}
(\partial_\mu R)(\epsilon,\mu) = \ & \lambda_2(\mu)(\lambda_1(\mu)-\lambda_3(\mu))\left((\lambda_2(\mu))^2-(\lambda_3(\mu))^2\right)\cdot \\
& \cdot\left(1-(\lambda_2(\mu))^2\right)^{-1}\left(1-(\lambda_3(\mu))^2\right)^{-1}\cdot \\
& \cdot[(A_1(\mu)+\epsilon^{-1}A_2(\mu))\exp(\epsilon^{-1}\lambda_1(\mu)) + \\
& \quad +(A_3(\mu)+\epsilon^{-1})\exp(\epsilon^{-1}\lambda_2(\mu)) + A_4(\mu)\exp(\epsilon^{-1}\lambda_3(\mu))],
\end{aligned}
$$

where the functions $A_i(\mu)$ $(1 \le i \le 4)$ are bounded in $W_{1/2}$. Let $K > 0$ be an upperbound for the absolute value of these functions in $W_{1/2}$. Then we have for ϵ sufficiently small:

$$
\begin{aligned}
&|[\ (A_1(\mu)+\epsilon^{-1}A_2(\mu))\exp(\epsilon^{-1}\lambda_1(\mu)) + \\
&\quad +(A_3(\mu)+\epsilon^{-1})\exp(\epsilon^{-1}\lambda_2(\mu)) + A_4(\mu)\exp(\epsilon^{-1}\lambda_3(\mu))]| \ge \\
&\ge (\epsilon^{-1}-K)\exp\left(\epsilon^{-1}\mathrm{Re}\,(\lambda_2(\mu))\right) + \\
&\quad -K(1+\epsilon^{-1})\exp\left(\epsilon^{-1}\mathrm{Re}\,(\lambda_1(\mu))\right) - K\exp\left(\epsilon^{-1}\mathrm{Re}\,(\lambda_3(\mu))\right) \ge \\
&\ge \exp\left(\epsilon^{-1}\mathrm{Re}\,(\lambda_2(\mu))\right)[(\epsilon^{-1}-K) - K(1+\epsilon^{-1})\exp(-\epsilon^{-1}) - Ke] > 0,
\end{aligned}
$$

where we have used the fact that $\mathrm{Re}\,(\lambda_1(\mu)) < -1$ and $\epsilon^{-1}\mathrm{Re}\,(\lambda_3(\mu) - \lambda_2(\mu)) < 1$. Thus the last inequality yields: $(\partial_\mu R)(\epsilon, \mu) \neq 0$ so that $(\partial_\mu D)(\epsilon, \mu) \neq 0$. □

Next, we need the following lemma:

Lemma 4.4 *Let ϵ_0 be such that for $\epsilon \in (0, \epsilon_0]$ the conclusions of the Lemmas 4.2 and 4.3 are valid. Then for every $(\epsilon', \mu') \in (0, \epsilon_0] \times \mathbf{C}$ with $D(\epsilon', \mu') = 0$ there is precisely one function $\mu : (0, \epsilon_0] \to \mathbf{C}$ which is the solution of the equation*

$$D\,(\epsilon, \mu(\epsilon)) = 0 \quad \forall \epsilon \in (0, \epsilon_0],$$

and takes value μ' for $\epsilon = \epsilon' : \mu(\epsilon') = \mu'$; moreover $\mu(\epsilon)$ is real analytic for $\epsilon \in (0, \epsilon_0]$.

Proof. Let $(\epsilon', \mu') \in (0, \epsilon_0] \times \mathbf{C}$ satisfy $D(\epsilon', \mu') = 0$. Then Lemmas 4.1 and 4.2 yield: $\mu' \in W_{\epsilon'}$. Thus as a consequence of Lemma 4.3 one has: $(\partial_\mu D)(\epsilon', \mu') \neq 0$. Therefore the implicit function theorem yields: there are constants $\delta_1, \delta_2 > 0$ and an analytic function $\mu : (\epsilon' - \delta_1, \epsilon' + \delta_2) \to \mathbf{C}$ such that

$$D\,(\epsilon, \mu(\epsilon)) = 0 \quad \forall \epsilon \in (\epsilon' - \delta_1, \epsilon' + \delta_2).$$

Suppose $\epsilon' - \delta_1 > 0$ and $|\mu(\epsilon)| \to \infty$ as $\epsilon \downarrow \epsilon' - \delta_1$. Then one has:

$$\begin{aligned}
(4.8)\quad & \exp\left(\epsilon^{-1}\left(\lambda_2\left(\mu(\epsilon)\right) - \lambda_3\left(\mu(\epsilon)\right)\right)\right) = \\
& = \lambda_1\left(\mu(\epsilon)\right)\left(\lambda_2\left(\mu(\epsilon)\right)\right)^{-1}\left(\lambda_2\left(\mu(\epsilon)\right) - \lambda_3\left(\mu(\epsilon)\right)\right) \cdot \\
& \quad \cdot \left(\lambda_1\left(\mu(\epsilon)\right) - \lambda_3\left(\mu(\epsilon)\right)\right)^{-1} \exp\left(\epsilon^{-1}\left(\lambda_1\left(\mu(\epsilon)\right) - \lambda_3\left(\mu(\epsilon)\right)\right)\right) + \\
& + \lambda_3\left(\mu(\epsilon)\right)\left(\lambda_2\left(\mu(\epsilon)\right)\right)^{-1}\left(\lambda_1\left(\mu(\epsilon)\right) - \lambda_2\left(\mu(\epsilon)\right)\right) \cdot \\
& \quad \cdot \left(\lambda_1\left(\mu(\epsilon)\right) - \lambda_3\left(\mu(\epsilon)\right)\right)^{-1}.
\end{aligned}$$

The right-hand side of (4.8) converges to $1/2 - (1/2)i\sqrt{3}$ as $\epsilon \downarrow \epsilon' - \delta_1$. (Here we use the fact that $\mu(\epsilon) \in W_\epsilon \subset W_{1/2}$ and thus $\mathrm{Re}\,(\lambda_1\,(\mu(\epsilon))) < -1$, $\mathrm{Re}\,(\lambda_3\,(\mu(\epsilon))) > 0$). Therefore $\lambda_2\,(\mu(\epsilon)) - \lambda_3\,(\mu(\epsilon))$ remains bounded as $\epsilon \downarrow \epsilon' - \delta_1$, in contradiction with the assumption that $|\mu(\epsilon)| \to \infty$ as $\epsilon \downarrow \epsilon' - \delta_1$. Thus there is a sequence $\epsilon_1, \epsilon_2, \cdots$ with $\epsilon_n \downarrow \epsilon' - \delta_1$ and $\mu(\epsilon_n) \to \mu_0 \in \mathbf{C}$, so that $D(\epsilon' - \delta_1, \mu_0) = 0$ and, as a consequence of Lemma 4.3, $(\partial_\mu D)(\epsilon' - \delta_1, \mu_0) \neq 0$. The implicit function theorem allows us to extend $\mu : (\epsilon' - \delta_1, \epsilon' + \delta_2) \to \mathbf{C}$ analytically up to the point $\epsilon' - \delta_1$. In this way μ may be extended to an analytic function on $(0, \epsilon_0]$, satisfying the required properties. This μ is uniquely determined as a consequence of the implicit function theorem. □

We shall also need the following lemma.

Lemma 4.5 *Let ϵ_0 be chosen as in Lemma 4.4. For every $n \in \mathbf{N}^*$ there exists an analytic function $\mu_n : (0, \epsilon_0] \to \mathbf{C}$ such that*

$$D(\epsilon, \mu_n(\epsilon)) = 0, \quad \forall \epsilon \in (0, \epsilon_0],$$

and

$$\mu_n(\epsilon) = (2/3) + n^2\pi^2\epsilon^2 - (4/3)n^2\pi^2\epsilon^3 + \\ + \left((5/36)n^4\pi^4 + (4/3)n^2\pi^2\right)\epsilon^4 + O(\epsilon^5) \tag{4.9}$$

as $\epsilon \downarrow 0$.

Proof. $F(\epsilon, \nu) := D(\epsilon, (2/3) + \epsilon^2\nu)$ is an entire function of ν for every $\epsilon > 0$. Using the formula

$$\lambda_i\left((2/3) + \epsilon^2\nu\right) - (1/3)\left(\lambda_i\left((2/3) + \epsilon^2\nu\right)\right)^3 = (2/3) + \epsilon^2\nu \quad (1 \le i \le 3),$$

we find for λ_i $(1 \le i \le 3)$ the following asymptotic formulae:

$$\lambda_1\left((2/3) + \epsilon^2\nu\right) = -2 - (1/3)\nu\epsilon^2 + O(\epsilon^4) \tag{4.10}$$

$$\lambda_2\left((2/3) + \epsilon^2\nu\right) = 1 + i\nu^{1/2}\epsilon + (1/6)\nu\epsilon^2 - i(5/72)\nu^{3/2}\epsilon^3 + O(\epsilon^4) \tag{4.11}$$

$$\lambda_3\left((2/3) + \epsilon^2\nu\right) = 1 - i\nu^{1/2}\epsilon + (1/6)\nu\epsilon^2 + i(5/72)\nu^{3/2}\epsilon^3 + O(\epsilon^4) \tag{4.12}$$

as $\epsilon \downarrow 0$, which hold uniformly on every compact set in $\mathbf{C}\backslash\overline{\mathbf{R}_-}$. Asymptotic formulae for $\partial_\epsilon^k \partial_\nu^l \left(\lambda_i\left((2/3) + \epsilon^2\nu\right)\right)$ $(k \in \mathbf{N}, l \in \mathbf{N})$ are obtained by differentiating the right-hand sides of (4.10)-(4.12) formally. Using (4.10)-(4.12), we get

$$\exp\left(\epsilon^{-1}\lambda_1\left((2/3) + \epsilon^2\nu\right)\right) = \\ = \exp(-2\epsilon^{-1})\left(1 - (1/3)\nu\epsilon + (1/18)\nu^2\epsilon^2 + O(\epsilon^3)\right) \tag{4.13}$$

$$\exp\left(\epsilon^{-1}\lambda_2\left((2/3) + \epsilon^2\nu\right)\right) = \exp(\epsilon^{-1})\exp(i\nu^{1/2})\cdot \\ \cdot\left(1 + (1/6)\nu\epsilon + \left((1/72)\nu^2 - i(5/72)\nu^{3/2}\right)\epsilon^2 + O(\epsilon^3)\right) \tag{4.14}$$

$$\exp\left(\epsilon^{-1}\lambda_3\left((2/3) + \epsilon^2\nu\right)\right) = \exp(\epsilon^{-1})\exp(-i\nu^{1/2})\cdot \\ \cdot\left(1 + (1/6)\nu\epsilon + \left((1/72)\nu^2 + i(5/72)\nu^{3/2}\right)\epsilon^2 + O(\epsilon^3)\right). \tag{4.15}$$

Writing

$$
\begin{aligned}
F(\epsilon,\nu) = \ & [\lambda_1\left((2/3)+\epsilon^2\nu\right)\left(\lambda_3\left((2/3)+\epsilon^2\nu\right)-\lambda_2\left((2/3)+\epsilon^2\nu\right)\right)\cdot \\
& \qquad \cdot\exp\left(\epsilon^{-1}\lambda_1\left((2/3)+\epsilon^2\nu\right)\right)+ \\
& +\lambda_2\left((2/3)+\epsilon^2\nu\right)\left(\lambda_1\left((2/3)+\epsilon^2\nu\right)-\lambda_3\left((2/3)+\epsilon^2\nu\right)\right)\cdot \\
& \qquad \cdot\exp\left(\epsilon^{-1}\lambda_2\left((2/3)+\epsilon^2\nu\right)\right)+ \\
& +\lambda_3\left((2/3)+\epsilon^2\nu\right)\left(\lambda_2\left((2/3)+\epsilon^2\nu\right)-\lambda_1\left((2/3)+\epsilon^2\nu\right)\right)\cdot \\
& \qquad \cdot\exp\left(\epsilon^{-1}\lambda_3\left((2/3)+\epsilon^2\nu\right)\right)]\cdot \\
& \cdot[\left(\lambda_3\left((2/3)+\epsilon^2\nu\right)-\lambda_2\left((2/3)+\epsilon^2\nu\right)\right)\cdot \\
& \cdot\left(\lambda_1\left((2/3)+\epsilon^2\nu\right)-\lambda_3\left((2/3)+\epsilon^2\nu\right)\right)\cdot \\
& \cdot\left(\lambda_2\left((2/3)+\epsilon^2\nu\right)-\lambda_1\left((2/3)+\epsilon^2\nu\right)\right)]^{-1}
\end{aligned}
$$

and substituting (4.10)-(4.15) into the last formula, we find:

(4.16) $F(\epsilon,\nu) =$

$$
\begin{aligned}
&= \epsilon^{-1}\exp(\epsilon^{-1})[\exp(-3\epsilon^{-1})\cdot \\
&\quad \cdot\left\{(2/9)\epsilon-(2/27)\nu\epsilon^2+\left((-5/81)\nu+(1/81)\nu^2\right)\epsilon^3+O(\epsilon^4)\right\}+ \\
&\quad -(1/3)\nu^{-1/2}\sin(\nu^{1/2})+ \\
&\quad +\left(-(1/18)\nu^{1/2}\sin(\nu^{1/2})-(2/9)\cos(\nu^{1/2})\right)\epsilon+ \\
&\quad +\Big(-(1/216)\nu^{3/2}\sin(\nu^{1/2})-(1/72)\nu\cos(\nu^{1/2})+ \\
&\qquad -(7/72)\nu^{1/2}\sin(\nu^{1/2})\Big)\epsilon^2+ \\
&\quad +O(\epsilon^3)] \text{ as } \epsilon\downarrow 0,
\end{aligned}
$$

uniformly with respect to ν on every compact subset in $\mathbf{C}\backslash\overline{\mathbf{R}_-}$, the asymptotic expansion still being valid after any number of differentiations of F with respect to ν and ϵ. Defining $Q(\epsilon,\nu) := \epsilon\exp(-\epsilon^{-1})F(\epsilon,\nu)$, we find that $Q(\epsilon,\nu)$ is a C^∞ function of $(\epsilon,\nu)\in[0,\infty)\times\mathbf{C}$ and that $Q(0,\nu) = -(1/3)\nu^{-1/2}\sin(\nu^{1/2})$. Since for $n\in\mathbf{N}^*$ one has: $Q(0,n^2\pi^2) = 0$ and $(\partial_\nu Q)(0,n^2\pi^2)\neq 0$, the implicit function theorem guarantees the existence of a $\delta_n > 0$ and a C^∞ function $\nu_n : [0,\delta_n)\to\mathbf{R}$ such that $Q\left(\epsilon,\nu_n(\epsilon)\right) = 0,\ \ \forall\epsilon\in[0,\delta_n)$. Moreover we have

$$\nu_n(\epsilon)\sim\sum_{k=0}^{\infty}a_k^n\epsilon^k$$

as $\epsilon \downarrow 0$, where the a_k^n can be computed explicitly by differentiating the relation $Q(\epsilon, \nu_n(\epsilon)) = 0$ a sufficient number of times with respect to ϵ and putting afterwards $\epsilon = 0$. Using (4.16), we find

$$\begin{aligned} a_0^n &= n^2\pi^2 \\ a_1^n &= -(4/3)n^2\pi^2 \\ a_2^n &= (5/36)n^4\pi^4 + (4/3)n^2\pi^2. \end{aligned}$$

Now $\mu_n(\epsilon) = (2/3) + \epsilon^2\nu_n(\epsilon)$ is a zero of $D(\epsilon,\mu)$: $D(\epsilon, \mu_n(\epsilon)) = 0, \quad \forall \epsilon \in (0, \delta_n)$. Lemma 4.4 allows us to extend $\mu_n(\epsilon)$ as an analytic function of $\epsilon \in (0, \epsilon_0]$ for every $n \in \mathbf{N}^*$. Moreover asymptotic expansion (4.9) is valid.

□

Finally the following auxiliary result will be needed as well.

Lemma 4.6 *Let ϵ_0 be chosen as in Lemma 4.4. Then the zeros $\mu_n(\epsilon), n = 1, 2, \ldots,$ of $D(\epsilon, \mu)$, $\epsilon \in (0, \epsilon_0]$, with the asymptotic expansions (4.9) are the only zeros of the function $D(\epsilon, \mu)$.*

Proof. Let $\epsilon' \in (0, \epsilon_0]$, $\mu' \in \mathbf{C}$ with $D(\epsilon', \mu') = 0$. Then Lemma 4.4 implies the existence of an analytic function $\mu : (0, \epsilon_0] \to \mathbf{C}$ such that

$$\begin{cases} D(\epsilon, \mu(\epsilon)) = 0 \quad \forall \epsilon \in (0, \epsilon_0] \\ \mu(\epsilon') = \mu'. \end{cases}$$

Thus (4.8) holds for every $\epsilon \in (0, \epsilon_0]$. Assume that there is a sequence $\{\epsilon_n\}, \epsilon_n \downarrow 0$ for $n \to \infty$, such that $|\mu(\epsilon_n)| \to \infty$, as $n \to \infty$.
Then one has: $|\lambda_2(\mu(\epsilon_n)) - \lambda_3(\mu(\epsilon_n))| \to \infty$ as $n \to \infty$.
Since $|\mathrm{Re}(\lambda_2(\mu(\epsilon_n)) - \lambda_3(\mu(\epsilon_n)))| < \epsilon_n \leq \epsilon_0$ we have:

$$\mathrm{Im}(\lambda_2(\mu(\epsilon_n)) - \lambda_3(\mu(\epsilon_n))) \to \infty \text{ as } n \to \infty.$$

Thus there is a sequence $\epsilon'_1 > \epsilon'_2 > \ldots > 0$ such that:

$$\begin{cases} \epsilon'_n \downarrow 0 \text{ as } n \to \infty, \\ (2\pi)^{-1}(\epsilon'_n)^{-1}\mathrm{Im}(\lambda_2(\mu(\epsilon'_n)) - \lambda_3(\mu(\epsilon'_n))) \in \mathbf{N}^*, \ \forall n \in \mathbf{N}^* \\ \mathrm{Im}(\lambda_2(\mu(\epsilon'_n)) - \lambda_3(\mu(\epsilon'_n))) \to \infty \text{ as } n \to \infty. \end{cases}$$

Therefore $|\mu(\epsilon'_n)| \to \infty$ as $n \to \infty$. Putting $\epsilon = \epsilon'_n$ in (4.8), one finds that the right-hand side of (4.8) converges to $(1/2) - (1/2)i\sqrt{3}$ as $n \to \infty$, whereas the left-hand side is real for every $n \in \mathbf{N}^*$, the latter giving rise to a contradiction.

Suppose $\epsilon_n \downarrow 0$ and $\mu(\epsilon_n) \to \mu$ as $n \to \infty$. Then using (4.8), one finds: $\mu = 2/3$, so that

$$\lim_{\epsilon \downarrow 0} \mu(\epsilon) = 2/3.$$

Again using (4.8) we find:

$$\lim_{\epsilon \downarrow 0} \exp\left(\epsilon^{-1}\left(\lambda_2(\mu(\epsilon)) - \lambda_3(\mu(\epsilon))\right)\right) = 1.$$

Thus there is a $k \in \mathbf{Z}$ such that

$$\lim_{\epsilon \downarrow 0} \epsilon^{-1}\left(\lambda_2(\mu(\epsilon)) - \lambda_3(\mu(\epsilon))\right) = 2k\pi i.$$

Therefore one has:

$$12 - 3\left(\lambda_1(\mu(\epsilon))\right)^2 = \left(\lambda_3(\mu(\epsilon)) - \lambda_2(\mu(\epsilon))\right)^2 = -4k^2\pi^2\epsilon^2 + o(\epsilon^2),$$

so that

$$\lambda_1(\mu(\epsilon)) = -2 - (1/3)k^2\pi^2\epsilon^2 + o(\epsilon^2)$$

and

$$\mu(\epsilon) = \lambda_1(\mu(\epsilon)) - (1/3)\left(\lambda_1(\mu(\epsilon))\right)^3 = (2/3) + k^2\pi^2\epsilon^2 + o(\epsilon^2).$$

Setting $\nu(\epsilon) := \epsilon^{-2}\left(\mu(\epsilon) - 2/3\right)$ we find

$$\begin{cases} Q(\epsilon, \nu(\epsilon)) = 0 \quad \forall \epsilon \in (0, \epsilon_0] \\ \lim_{\epsilon \downarrow 0} \nu(\epsilon) = k^2\pi^2, \end{cases}$$

where $Q(\epsilon, \nu)$ is defined in the proof of Lemma 4.5. Hence one has: $Q(0, k^2\pi^2) = 0$, so that $k \neq 0$. The implicit function theorem implies: $\nu(\epsilon) = \nu_k(\epsilon) \ \ \forall \epsilon \in (0, \epsilon_0]$, where $\nu_k(\epsilon)$ was defined in the proof of Lemma 4.5. Therefore $\mu(\epsilon) = \mu_k(\epsilon), \ \ \forall \epsilon \in (0, \epsilon_0]$, too.

Theorem 4.1 *Let ϵ_0 be sufficiently small. For $\epsilon \in (0, \epsilon_0]$ all the eigenvalues $\lambda_n(\epsilon)$ of (4.1), (4.2) are strictly positive and have the form:*

$$\lambda_n(\epsilon) = \epsilon^{-1}\mu_n(\epsilon), \quad n = 1, 2, \ldots,$$

where $\mu_n(\epsilon)$ for each given $n \in \mathbf{N}^*$ have the asymptotic expansions (4.9) and for $\lambda_n(\epsilon), n \in \mathbf{N}^*$, one has the asymptotic formulae:

$$(4.17) \quad \lambda_n(\epsilon) = (2/3)\epsilon^{-1} + n^2\pi^2\epsilon - (4/3)n^2\pi^2\epsilon^2 + \\ + \left((5/36)n^4\pi^4 + (4/3)n^2\pi^2\right)\epsilon^3 + O(\epsilon^4) \ \textit{as} \ \epsilon \downarrow 0.$$

The normalized eigenfunctions u_n^ϵ, $\|u_n^\epsilon\|_{L^2(U)} = 1$, associated with the eigenvalues $\lambda_n(\epsilon)$, for each given $n \in \mathbf{N}^*$ have the following asymptotic expansions uniformly with respect to $x \in \overline{U}$:

$$(4.18)\quad u_n^\epsilon(x) = \gamma_n \epsilon^{-3/2} \exp\left(-\epsilon^{-1}(1-x)\right)(\sin \pi n x + O(\epsilon)) \quad \textit{as } \epsilon \downarrow 0,$$

where $\gamma_n = 2 \cdot 5^{-1/2} \cdot (n\pi)^{-1}$.

Proof. Asymptotic expansion (4.17) has already been proved. Using (4.4) we note that the eigenfunction associated with the eigenvalue $\lambda_n(\epsilon)$ is a scalar multiple of

$$\begin{aligned}(4.19)\quad v_n^\epsilon(x) = \ & (\lambda_3(\mu_n(\epsilon)) - \lambda_2(\mu_n(\epsilon))) \exp(\epsilon^{-1}\lambda_1(\mu_n(\epsilon))x) + \\ & + (\lambda_1(\mu_n(\epsilon)) - \lambda_3(\mu_n(\epsilon))) \exp(\epsilon^{-1}\lambda_2(\mu_n(\epsilon))x) + \\ & + (\lambda_2(\mu_n(\epsilon)) - \lambda_1(\mu_n(\epsilon))) \exp(\epsilon^{-1}\lambda_3(\mu_n(\epsilon))x).\end{aligned}$$

We have the following asymptotic formulae:

$$\begin{aligned}(4.20)\quad \lambda_1(\mu_n(\epsilon)) = \ & -2 - (1/3)n^2\pi^2\epsilon^2 + (4/9)n^2\pi^2\epsilon^3 + \\ & + \left((1/36)n^4\pi^4 - (4/9)n^2\pi^2\right)\epsilon^4 + O(\epsilon^5),\end{aligned}$$

$$\begin{aligned}(4.21)\quad \lambda_2(\mu_n(\epsilon)) = \ & 1 + in\pi\epsilon + \left(-(2/3)in\pi + (1/6)n^2\pi^2\right)\epsilon^2 + \\ & + \left((4/9)in\pi - (2/9)n^2\pi^2\right)\epsilon^3 + O(\epsilon^4),\end{aligned}$$

$$\begin{aligned}(4.22)\quad \lambda_3(\mu_n(\epsilon)) = \ & 1 - in\pi\epsilon + \left((2/3)in\pi + (1/6)n^2\pi^2\right)\epsilon^2 + \\ & + \left(-(4/9)in\pi - (2/9)n^2\pi^2\right)\epsilon^3 + O(\epsilon^4).\end{aligned}$$

Using (4.19)-(4.22) and the fact that $\lambda_1(\mu_n(\epsilon)) \in \mathbf{R}, \lambda_2(\mu_n(\epsilon)) = \overline{\lambda_3(\mu_n(\epsilon))}$ $\forall \epsilon \in (0, \epsilon_0]$, we get:

$$\begin{aligned}\|v_n^\epsilon\|^2_{L^2(U)} & = \int_U v_n^\epsilon(x) \cdot \overline{v_n^\epsilon(x)} dx = \\ & = -(1/2)\epsilon(\lambda_3(\mu_n(\epsilon)) - \lambda_2(\mu_n(\epsilon)))^2 (\lambda_1(\mu_n(\epsilon)))^{-1} \cdot \\ & \qquad \cdot (\exp(2\epsilon^{-1}\lambda_1(\mu_n(\epsilon))) - 1) + \\ & \quad -2\epsilon(\lambda_3(\mu_n(\epsilon)) - \lambda_2(\mu_n(\epsilon)))(\lambda_1(\mu_n(\epsilon)) - \lambda_2(\mu_n(\epsilon))) \cdot \\ & \qquad \cdot (\lambda_2(\mu_n(\epsilon)))^{-1} (\exp(-\epsilon^{-1}\lambda_2(\mu_n(\epsilon))) - 1) +\end{aligned}$$

$$
\begin{aligned}
&-2\epsilon\left(\lambda_3(\mu_n(\epsilon))-\lambda_2(\mu_n(\epsilon))\right)\left(\lambda_3(\mu_n(\epsilon))-\lambda_1(\mu_n(\epsilon))\right)\cdot\\
&\qquad\cdot\left(\lambda_3(\mu_n(\epsilon))\right)^{-1}\left(\exp\left(-\epsilon^{-1}\lambda_3(\mu_n(\epsilon))\right)-1\right)+\\
&-2\epsilon\left(\lambda_1(\mu_n(\epsilon))-\lambda_3(\mu_n(\epsilon))\right)\left(\lambda_1(\mu_n(\epsilon))-\lambda_2(\mu_n(\epsilon))\right)\cdot\\
&\qquad\cdot\left(\lambda_1(\mu_n(\epsilon))\right)^{-1}\left(\exp\left(-\epsilon^{-1}\lambda_1(\mu_n(\epsilon))\right)-1\right)+\\
&-(1/2)\epsilon\left(\lambda_1(\mu_n(\epsilon))-\lambda_3(\mu_n(\epsilon))\right)^2\left(\lambda_2(\mu_n(\epsilon))\right)^{-1}\cdot\\
&\qquad\cdot\left(\exp\left(2\epsilon^{-1}\lambda_2(\mu_n(\epsilon))\right)-1\right)+\\
&-(1/2)\epsilon\left(\lambda_2(\mu_n(\epsilon))-\lambda_1(\mu_n(\epsilon))\right)^2\left(\lambda_3(\mu_n(\epsilon))\right)^{-1}\cdot\\
&\qquad\cdot\left(\exp\left(2\epsilon^{-1}\lambda_3(\mu_n(\epsilon))\right)-1\right)\\
\sim\ &-2\epsilon\left(\lambda_1(\mu_n(\epsilon))-\lambda_3(\mu_n(\epsilon))\right)\left(\lambda_1(\mu_n(\epsilon))-\lambda_2(\mu_n(\epsilon))\right)\cdot\\
&\qquad\cdot\left(\lambda_1(\mu_n(\epsilon))\right)^{-1}\exp\left(-\epsilon^{-1}\lambda_1(\mu_n(\epsilon))\right)+\\
&-(1/2)\epsilon\left(\lambda_1(\mu_n(\epsilon))-\lambda_3(\mu_n(\epsilon))\right)^2\left(\lambda_2(\mu_n(\epsilon))\right)^{-1}\cdot\\
&\qquad\cdot\exp\left(2\epsilon^{-1}\lambda_2(\mu_n(\epsilon))\right)+\\
&-(1/2)\epsilon\left(\lambda_2(\mu_n(\epsilon))-\lambda_1(\mu_n(\epsilon))\right)^2\left(\lambda_3(\mu_n(\epsilon))\right)^{-1}\cdot\\
&\qquad\cdot\exp\left(2\epsilon^{-1}\lambda_3(\mu_n(\epsilon))\right)=\\
=\ &\exp(2\epsilon^{-1})\epsilon^3\left(45n^2\pi^2+O(\epsilon)\right).
\end{aligned}
$$

Thus

$$\|v_n^\epsilon\|_{L^2(U)}=\exp(\epsilon^{-1})\epsilon^{3/2}\left(3\sqrt{5}n\pi+O(\epsilon)\right).$$

Substituting (4.20)-(4.22) into (4.19) we find:

$$
\begin{aligned}
v_n^\epsilon(x)\ &=\ \left(-2in\pi\epsilon+O(\epsilon^2)\right)\exp(-2\epsilon^{-1}x)\left(1+O(\epsilon)\right)+\\
&\qquad+\left(-3+O(\epsilon)\right)\exp(\epsilon^{-1}x)\exp(in\pi x)\left(1+O(\epsilon)\right)+\\
&\qquad+\left(3+O(\epsilon)\right)\exp(\epsilon^{-1}x)\exp(-in\pi x)\left(1+O(\epsilon)\right)=\\
&=\exp(\epsilon^{-1}x)\left\{\exp(-3\epsilon^{-1}x)\left(-2in\pi\epsilon+O(\epsilon^2)\right)+\right.\\
&\qquad\left.-6i\sin\pi nx+O(\epsilon)\right\}\ \text{ as } \epsilon\downarrow 0,
\end{aligned}
$$

uniformly with respect to $x\in\overline{U}$. Setting $u_n^\epsilon(x)=i\left(\|v_n^\epsilon\|_{L^2(U)}\right)^{-1}v_n^\epsilon(x)$ we find for u_n^ϵ asymptotic formula (4.18). □

On the half-line $\mathbf{R}_+ := \{x_1 > 0\}$ the mixed initial-boundary value problem for the operator $\partial_{x_0} + \partial_{x_1} + \epsilon^2\partial^3_{x_1}$ is not well-posed uniformly with respect to $\epsilon \in (0, \epsilon_0]$ ([4]). On a finite interval $x_1 \in U \subset \mathbf{R}_+$ one can state well-posed mixed problems for this operator. However, for its stationary part, which is no longer an elliptic singular perturbation, one comes across of the phenomenon of highly oscillatory solutions, as $\epsilon \downarrow 0$, the boundary layer phenomenon being no more characteristic for this non-elliptic singular perturbation.

The asymptotic formula (4.17) (and, especially the fact that $\lambda_n(\epsilon) > 0, \forall n \in \mathbf{N}^*, \forall \epsilon \in (0, \epsilon_0]$, ϵ_0 sufficiently small) for the elliptic singular perturbation $\partial_{x_1} - \epsilon^2\partial^3_{x_1}$, shows that for the solutions of the corresponding mixed problem on $U = (0,1)$ for the operator $\partial_{x_0} + \partial_{x_1} - \epsilon^2\partial^3_{x_1}$, the dissipation of energy takes place; thus this operator on a finite interval is, in fact, a dissipative singular perturbation.

Now we shall consider the eigenvalue problem for the non-elliptic singular perturbation $\partial_{x_1} + \epsilon^2\partial^3_{x_1}$ and show that the corresponding non-stationary operator $\partial_{x_0} + \partial_{x_1} + \epsilon^2\partial^3_{x_1}$ is no longer a purely dissipative singular perturbation on a finite interval $x_1 \in U \subset \mathbf{R}_+$.

More precisely, we shall show that there exists a sequence $\{\epsilon_p\}_{p=1}^{\infty}$, $\epsilon_p \to 0$ as $p \to \infty$ such that the operator $\partial_{x_1} + \epsilon^2\partial^3_{x_1}$ on U with corresponding boundary conditions on $\partial U = \{0, 1\}$ has purely imaginary eigenvalues $\lambda_n(\epsilon_p), 1 \le n \le n_p$, where $n_p \to \infty$ as $\epsilon_p \to 0$.

Thus consider the singularly perturbed eigenvalue problem:

$$-(\partial_x + 3\epsilon^2\partial^3_x)u(x) = \lambda u(x), \quad x \in U \tag{4.23}$$

$$u(0) = u(1) = (\partial_x u)(0) = 0. \tag{4.24}$$

Here the factor 3 is chosen for the sake of convenience, the minus sign has to be chosen since eigenvalue problem (4.23), (4.24) appears to have only eigenvalues with non-negative real part. Therefore the mixed initial-boundary value problem corresponding to the operator $\partial_{x_0} - \partial_{x_1} - \epsilon^2\partial^3_{x_1}$ has to be considered for $x_0 \ge 0$.

As in [5] we use the substitution $u(x) = \exp\left((1/3)i\epsilon^{-1}x\right) v(x)$ in (4.23), (4.24), which yields:

$$\begin{cases} -(\epsilon\partial_x + i)(\partial^2_x v)(x) = (1/3)\epsilon^{-1}(\lambda + (2/9)i\epsilon^{-1})v(x), x \in U \\ v(0) = v(1) = (\partial_x v)(0) = 0. \end{cases}$$

Therefore we consider the following singularly perturbed eigenvalue problem:

$$(4.25) \qquad -(\epsilon\partial_x + i)(\partial_x^2 u)(x) = \mu u(x), \quad x \in U = (0,1)$$

$$(4.24) \qquad u(0) = u(1) = (\partial_x u)(0) = 0.$$

The solution of (4.25) satisfying the initial conditions $u(0) = (\partial_x u)(0) = 0$, $(\partial_x^2 u)(0) = 1$ is given by the following formula:

$$(4.26) \qquad \begin{aligned} u(x) = [&(\lambda_3(\epsilon,\mu) - \lambda_2(\epsilon,\mu))\exp(\lambda_1(\epsilon,\mu)x) + \\ &+ (\lambda_1(\epsilon,\mu) - \lambda_3(\epsilon,\mu))\exp(\lambda_2(\epsilon,\mu)x) + \\ &+ (\lambda_2(\epsilon,\mu) - \lambda_1(\epsilon,\mu))\exp(\lambda_3(\epsilon,\mu)x)]\cdot \\ \cdot[&(\lambda_3(\epsilon,\mu) - \lambda_2(\epsilon,\mu))(\lambda_3(\epsilon,\mu) - \lambda_1(\epsilon,\mu))(\lambda_2(\epsilon,\mu) - \lambda_1(\epsilon,\mu))]^{-1}, \end{aligned}$$

where the $\lambda_i(\epsilon,\mu)(1 \le i \le 3)$ are the three zeros of the equation

$$(4.27) \qquad \epsilon\lambda^3 + i\lambda^2 + \mu = 0.$$

Substitution of $x = 1$ in (4.26) yields: μ is an eigenvalue of (4.25), (4.24) iff μ is a zero of the function

$$(4.28) \qquad \begin{aligned} \Phi(\epsilon,\mu) = [&(\lambda_3(\epsilon,\mu) - \lambda_2(\epsilon,\mu))\exp(\lambda_1(\epsilon,\mu)) + \\ &+ (\lambda_1(\epsilon,\mu) - \lambda_3(\epsilon,\mu))\exp(\lambda_2(\epsilon,\mu)) + \\ &+ (\lambda_2(\epsilon,\mu) - \lambda_1(\epsilon,\mu))\exp(\lambda_3(\epsilon,\mu))]\cdot \\ \cdot[&(\lambda_3(\epsilon,\mu) - \lambda_2(\epsilon,\mu))(\lambda_3(\epsilon,\mu) - \lambda_1(\epsilon,\mu))(\lambda_2(\epsilon,\mu) - \lambda_1(\epsilon,\mu))]^{-1}, \end{aligned}$$

which is for every $\epsilon \neq 0$, an entire function of μ.

Lemma 4.7 *If* $\epsilon > 0, \mu \in \mathbf{C}, \Phi(\epsilon,\mu) = 0$ *then* $Re\,\mu \ge 0$. *Moreover if* $(\epsilon,\mu) \in (0,\infty) \times \mathbf{C}$ *is a zero of* $\Phi(\epsilon,\mu)$ *and* $Re\,\mu = 0$, *then the corresponding eigenfunction* u *of* (4.25), (4.24) *satisfies the extra boundary condition:* $(\partial_x u)(1) = 0$.

Proof. Let $\epsilon > 0, \mu = \alpha + i\beta$ be a zero of $\Phi(\epsilon,\mu)$. Then there is a non-trivial solution of the problem:

$$\begin{cases} -(\epsilon\partial_x + i)\partial_x^2 u = \mu u, \; x \in U \\ u(0) = u(1) = (\partial_x u)(0) = 0. \end{cases}$$

Writing $u = v + iw$, the real-valued functions v, w satisfy the system

$$(4.29) \qquad -\epsilon\partial_x^3 v + \partial_x^2 w = \alpha v - \beta w$$

$$(4.30) \qquad -\epsilon\partial_x^3 w - \partial_x^2 v = \beta v + \alpha w$$

and the boundary conditions

$$v(0) = v(1) = (\partial_x v)(0) = w(0) = w(1) = (\partial_x w)(0) = 0.$$

Multiplying (4.29) by v, (4.30) by w we get after partial integration:

$$\begin{aligned}\alpha\int_U (v^2 + w^2)dx &= \int_U [-\epsilon(v\partial_x^3 v + w\partial_x^3 w) + (v\partial_x^2 w - w\partial_x^2 v)]dx = \\ &= \epsilon\int_U (\partial_x^2 v\partial_x v + \partial_x^2 w\partial_x w)dx = \\ &= (1/2)\epsilon\left(((\partial_x v)(1))^2 + ((\partial_x w)(1))^2\right) \geq 0.\end{aligned}$$

Thus: $\alpha \geq 0$ and $\alpha = 0$ implies: $(\partial_x u)(1) = 0$. □

The eigenvalues of the reduced problem

$$\begin{cases} -i\partial_x^2 u = \mu u \\ u(0) = u(1) = 0 \end{cases}$$

are $\mu_k^0 = ik^2\pi^2 (k \in \mathbf{N}\setminus\{0\})$ and the corresponding eigenfunctions are $u_k^0 = C\sin k\pi x$. Choosing $k \in \mathbf{N}\setminus\{0\}, \rho_k > 0$ such that the circle $\Gamma_k = \{z \in \mathbf{C} \mid |z - ik^2\pi^2| = \rho_k\}$ precisely encloses one eigenvalue of the reduced problem, namely $\mu_k^0 = ik^2\pi^2$, we will prove that for ϵ sufficiently small, $\Phi(\epsilon,\mu)$ has exactly one zero μ_k^ϵ in the disk $\Delta_k = \{z \in \mathbf{C} \mid |z - ik^2\pi^2| < \rho_k\}$. Moreover we will give an asymptotic formula for μ_k^ϵ. Using (4.27) the three zeros $\lambda_i(\epsilon,\mu)\,(1 \leq i \leq 3)$ are found to satisfy the following asymptotic formulae as $\epsilon \downarrow 0$:

(4.31) $$\lambda_1(\epsilon,\mu) = -i\epsilon^{-1} + \mu\epsilon + O(\epsilon^3),$$

(4.32) $$\lambda_2(\epsilon,\mu) = i^{1/2}\mu^{1/2} - (1/2)\mu\epsilon + (5/8)i^{-1/2}\mu^{3/2}\epsilon^2 + O(\epsilon^3),$$

(4.33) $$\lambda_3(\epsilon,\mu) = -i^{1/2}\mu^{1/2} - (1/2)\mu\epsilon - (5/8)i^{-1/2}\mu^{3/2}\epsilon^2 + O(\epsilon^3),$$

where $i^{1/2} = \exp(i\pi/4)$ and the branch $\mu^{1/2}$ is chosen so that $(ik^2\pi^2)^{1/2} = i^{1/2}k\pi$. The terms $O(\epsilon^3)$ are uniformly with respect to $\mu \in \overline{\Delta_k}$. Substitution of (4.31)-(4.33) into (4.28) yields:

(4.34) $$\begin{aligned}\Psi(\epsilon,\mu) &:= \epsilon^{-1}\Phi(\epsilon,\mu) = \\ &= \epsilon^{-1}[\left(-2i^{1/2}\mu^{1/2} - (5/4)i^{-1/2}\mu^{3/2}\epsilon^2 + O(\epsilon^3)\right)\cdot \\ &\qquad \cdot\exp(-i\epsilon^{-1})\,(1 + \mu\epsilon + O(\epsilon^2)) +\end{aligned}$$

$$
\begin{aligned}
&+\left(-i\epsilon^{-1}+i^{1/2}\mu^{1/2}+(3/2)\mu\epsilon+O(\epsilon^2)\right)\exp(i^{1/2}\mu^{1/2})\cdot \\
&\qquad\cdot\left(1-(1/2)\mu\epsilon+\left((5/8)i^{-1/2}\mu^{3/2}+(1/8)\mu^2\right)\epsilon^2+O(\epsilon^3)\right)+ \\
&+\left(i\epsilon^{-1}+i^{1/2}\mu^{1/2}-(3/2)\mu\epsilon+O(\epsilon^2)\right)\exp(-i^{1/2}\mu^{1/2})\cdot \\
&\qquad\cdot\left(1-(1/2)\mu\epsilon+\left(-(5/8)i^{-1/2}\mu^{3/2}+(1/8)\mu^2\right)\epsilon^2+O(\epsilon^3)\right)]\cdot \\
&\cdot\epsilon^2\left(2i^{1/2}\mu^{1/2}-(27/4)i^{-1/2}\mu^{3/2}\epsilon^2+O(\epsilon^3)\right)^{-1}= \\
&=i^{-1/2}\mu^{-1/2}\sin(i^{-1/2}\mu^{1/2})+ \\
&+\epsilon\left(-(1/2)i^{-1/2}\mu^{1/2}\sin(i^{-1/2}\mu^{1/2})+\cos(i^{-1/2}\mu^{1/2})\right)+ \\
&+\epsilon^2\left(-(15/8)i^{1/2}\mu^{1/2}\sin(i^{-1/2}\mu^{1/2})-(9/8)\mu\cos(i^{-1/2}\mu^{1/2})+\right. \\
&\qquad\left.+(1/8)i^{-1/2}\mu^{3/2}\sin(i^{-1/2}\mu^{1/2})\right)+ \\
&+O(\epsilon^3)+ \\
&+\exp(-i\epsilon^{-1})\left(-\epsilon-\mu\epsilon^2+O(\epsilon^3)\right),
\end{aligned}
$$

where the terms $O(\epsilon^3)$ are uniformly with respect to $\mu\in\overline{\Delta_k}$. Since the function $\Psi(0,\mu)=i^{-1/2}\mu^{-1/2}\sin(i^{-1/2}\mu^{1/2})$ does not vanish on Γ_k, Rouché's theorem implies for ϵ sufficiently small that $\Phi(\epsilon,\mu)$ precisely has one zero μ_k^ϵ in Δ_k. In order to get an asymptotic formula for μ_k^ϵ we proceed in the following way.
Introduce:

$$
\begin{aligned}
G(\epsilon,z,\mu):=\epsilon^{-1}\;&[(\lambda_3(\epsilon,\mu)-\lambda_2(\epsilon,\mu))\exp\left(\lambda_1(\epsilon,\mu)+i\epsilon^{-1}\right)z+ \\
&+(\lambda_1(\epsilon,\mu)-\lambda_3(\epsilon,\mu))\exp\left(\lambda_2(\epsilon,\mu)\right)+ \\
&+(\lambda_2(\epsilon,\mu)-\lambda_1(\epsilon,\mu))\exp\left(\lambda_3(\epsilon,\mu)\right)]\cdot \\
&\cdot[(\lambda_3(\epsilon,\mu)-\lambda_2(\epsilon,\mu))(\lambda_3(\epsilon,\mu)-\lambda_1(\epsilon,\mu))(\lambda_2(\epsilon,\mu)-\lambda_1(\epsilon,\mu))]^{-1}.
\end{aligned}
$$

Note that $\Psi(\epsilon,\mu)=G(\epsilon,\exp(-i\epsilon^{-1}),\mu)$. For ϵ sufficiently small (ϵ may be complex) and $\mu\in\Delta_k$ equation (4.27) has no multiple zeros. Therefore using (4.31)-(4.33), one comes to the conclusion that $\lambda_1(\epsilon,\mu)+i\epsilon^{-1}$, $\lambda_2(\epsilon,\mu)$ and $\lambda_3(\epsilon,\mu)$ are analytic functions of $(\epsilon,\mu)\in\{\epsilon\in\mathbf{C}\mid|\epsilon|<\epsilon'\}\times\Delta_k$, provided that ϵ' is sufficiently small. Thus $G(\epsilon,z,\mu)$ is an analytic function of $(\epsilon,z,\mu)\in\{\epsilon\in\mathbf{C}\mid|\epsilon|<\epsilon'\}\times S^1\times\Delta_k$. Since

$$G(0,z,\mu)=i^{-1/2}\mu^{-1/2}\sin(i^{-1/2}\mu^{1/2})$$

and

$$\begin{aligned}(\partial_\mu G)(0,z,\mu) = \; & -(1/2)i^{-1/2}\mu^{-3/2}\sin(i^{-1/2}\mu^{1/2})+ \\ & +(1/2)i^{-1}\mu^{-1}\cos(i^{-1/2}\mu^{1/2}) \;\; \forall \mu \in \Delta_k, \forall z \in S^1,\end{aligned}$$

we find:

$$G(0,z,ik^2\pi^2) = 0 \;\; \forall z \in S^1,$$

$$(\partial_\mu G)(0,z,ik^2\pi^2) \neq 0 \;\; \forall z \in S^1.$$

The implicit function theorem implies: there is an $\epsilon_k > 0$ and an analytic function $\mu_k : \{\epsilon \in \mathbf{C} \mid |\epsilon| < \epsilon_k\} \times S^1 \to \mathbf{C}$ such that

$$\begin{cases} G(\epsilon, z, \mu_k(\epsilon, z)) = 0, & \forall \epsilon \in \{\epsilon \in \mathbf{C} \mid |\epsilon| < \epsilon_k\}, \forall z \in S^1 \\ \mu_k(0,z) = ik^2\pi^2, & \forall z \in S^1 \end{cases}$$

and the following asymptotic formula holds, uniformly with respect to $z \in S^1$:

$$\mu_k(\epsilon, z) = \mu_k(0,z) + \epsilon(\partial_\epsilon \mu_k)(0,z) + O(\epsilon^2).$$

Differentiation of the relation $G(\epsilon, z, \mu_k(\epsilon, z)) = 0$ with respect to ϵ yields:

$$(\partial_\epsilon G)(0,z,ik^2\pi^2) + (\partial_\mu G)(0,z,ik^2\pi^2)(\partial_\epsilon \mu_k)(0,z) = 0.$$

Using (4.34) we get:

$$(\partial_\epsilon G)(0,z,ik^2\pi^2) = (-1)^k - z$$

$$(\partial_\mu G)(0,z,ik^2\pi^2) = (1/2)i^{-1}(ik^2\pi^2)^{-1}(-1)^k.$$

Hence:

$$(\partial_\epsilon \mu_k)(0,z) = 2k^2\pi^2\left(1 - (-1)^k z\right).$$

Therefore:

$$\mu_k(\epsilon, z) = ik^2\pi^2 + 2k^2\pi^2\left(1 - (-1)^k z\right)\epsilon + O(\epsilon^2),$$

and

$$\begin{aligned}\mu_k^\epsilon &= \mu_k\left(\epsilon, \exp(-i\epsilon^{-1})\right) = \\ &= ik^2\pi^2 + 2k^2\pi^2\left(1 - (-1)^k \exp(-i\epsilon^{-1})\right)\epsilon + O(\epsilon^2)\end{aligned} \tag{4.35}$$

as $\epsilon \downarrow 0$. We have proved the following result:

Theorem 4.2 *For every $k \in \mathbf{N}^*$ there are constants $\rho_k > 0, \epsilon_k > 0$ such that eigenvalue problem* (4.25), (4.24) *exactly has one eigenvalue μ_k^ϵ satisfying $|\mu_k^\epsilon - ik^2\pi^2| < \rho_k$ for every $\epsilon \in (0, \epsilon_k]$. Moreover asymptotic formula* (4.35) *holds for each given $k \in \mathbf{N}^*$.*

We already noted that λ is an eigenvalue of (4.23), (4.24) iff $(1/3)\epsilon^{-1}(\lambda + (2/9)i\epsilon^{-1})$ is an eigenvalue of (4.25), (4.24). Since the differential operator in (4.23) has real coefficients for $\epsilon > 0$ we also have: λ is an eigenvalue of (4.23), (4.24) iff $\overline{\lambda}$ is an eigenvalue of (4.23), (4.24). Therefore the following statements are equivalent:

(i) μ is an eigenvalue of (4.25), (4.24);

(ii) $-(2/9)i\epsilon^{-1} + 3\epsilon\mu$ is an eigenvalue of (4.23), (4.24);

(iii) $(2/9)i\epsilon^{-1} + 3\epsilon\overline{\mu}$ is an eigenvalue of (4.23),(4.24);

(iv) $(4/27)i\epsilon^{-2} + \overline{\mu}$ is an eigenvalue of (4.25), (4.24).

Thus with k, ϵ_k and μ_k^ϵ as in Theorem 4.2, one has:

$$\tau_k^\epsilon := (4/27)i\epsilon^{-2} + \overline{\mu_k^\epsilon} \tag{4.36}$$

is an eigenvalue of (4.25), (4.24) for $0 < \epsilon < \epsilon_k$ admitting the asymptotic formula

$$\tau_k^\epsilon = (4/27)i\epsilon^{-2} - ik^2\pi^2 + 2k^2\pi^2\left(1 - (-1)^k \exp(i\epsilon^{-1})\right)\epsilon + O(\epsilon^2) \text{ as } \epsilon \downarrow 0.$$

Now we are going to prove that with $\mu_k^\epsilon, \tau_k^\epsilon\,(k \in \mathbf{N}^*)$ defined as above, we have all the eigenvalues of (4.25), (4.24), i.e. if $\mu : (0, \epsilon_0] \to \mathbf{C}$ is a continuous function satisfying $\Phi\,(\epsilon, \mu(\epsilon)) = 0$ for every $\epsilon \in (0, \epsilon_0]$, then there is a $k \in \mathbf{N}^*$ such that $\mu(\epsilon) = \mu_k^\epsilon$ or $\mu(\epsilon) = \tau_k^\epsilon$, for ϵ sufficiently small. In order to do so we need some more information about the zeros $\lambda_i(\epsilon, \mu)\,(1 \le i \le 3)$ of (4.27).

Multiplying (4.27) by ϵ^2 and setting $\nu = \epsilon^2\mu - (2/27)i$ we find

$$\epsilon\lambda_j(\epsilon, \mu) + i/3 = \rho_j(\nu) \quad (1 \le j \le 3),$$

where $\rho_j(\nu)$ $(1 \le j \le 3)$ are the three zeros of

$$\rho^3 + (1/3)\rho + \nu = 0.$$

Now (4.28) becomes:

$$\begin{aligned} H(\epsilon, \nu) &:= \epsilon^{-2} \exp\left((1/3)i\epsilon^{-1}\right) \Phi(\epsilon, \mu) = \\ &= \left((\rho_3(\nu) - \rho_2(\nu)) \exp\left(\epsilon^{-1}\rho_1(\nu)\right) + \right. \\ &\quad + (\rho_1(\nu) - \rho_3(\nu)) \exp\left(\epsilon^{-1}\rho_2(\nu)\right) + \\ &\quad \left. + (\rho_2(\nu) - \rho_1(\nu)) \exp\left(\epsilon^{-1}\rho_3(\nu)\right)\right) \cdot \\ &\quad \cdot \left((\rho_3(\nu) - \rho_2(\nu))(\rho_3(\nu) - \rho_1(\nu))(\rho_2(\nu) - \rho_1(\nu))\right)^{-1}. \end{aligned} \tag{4.37}$$

The equation $\rho^3 + (1/3)\rho + \nu = 0$ has multiple zeros for $\nu = \pm(2/27)i$. For $\nu = i\alpha, -2/27 < \alpha < 2/27$ this equation has three purely imaginary zeros. We need the following lemma:

Lemma 4.8 *The function* $H(\epsilon, \nu)$ *defined by* (4.28), (4.37) *does not vanish, if* $\nu = i\alpha, \alpha \in \mathbf{R}\backslash(-2/27, 2/27), \epsilon > 0$.

Proof. For $\alpha > 2/27$ ($\alpha < -2/27$, respectively) the roots $\rho_i(\nu)$ may be written in the following form:

$$\rho_1(\nu) = ip, \tag{4.38}$$

$$\rho_2(\nu) = -(1/2)(3p^2 - 4/3)^{1/2} - (1/2)ip, \tag{4.39}$$

$$\rho_3(\nu) = (1/2)(3p^2 - 4/3)^{1/2} - (1/2)ip, \tag{4.40}$$

where $p > 2/3$ ($p < -2/3$ respectively) is the unique real solution of

$$p^3 - (1/3)p = \alpha.$$

Substitution of (4.38)-(4.40) in (4.37) yields:

$$\begin{aligned}
&H(\epsilon, \nu) = \\
&= -\Big((3p^2 - 4/3)^{1/2}\exp(i\epsilon^{-1}p) + \\
&\quad + \Big(-(1/2)(3p^2 - 4/3)^{1/2} + (3/2)ip\Big)\cdot \\
&\qquad \cdot\exp(-(1/2)\epsilon^{-1}(3p^2 - 4/3)^{1/2} - (1/2)i\epsilon^{-1}p) + \\
&\quad + \Big(-(1/2)(3p^2 - 4/3)^{1/2} - (3/2)ip\Big)\cdot \\
&\qquad \cdot\exp((1/2)\epsilon^{-1}(3p^2 - 4/3)^{1/2} - (1/2)i\epsilon^{-1}p)\Big)\cdot \\
&\quad \cdot\Big((3p^2 - 1/3)(3p^2 - 4/3)^{1/2}\Big)^{-1} = \\
&= -\Big((3p^2 - 1/3)(3p^2 - 4/3)^{1/2}\Big)^{-1}\exp(-(1/2)i\epsilon^{-1}p)\cdot \\
&\quad \cdot\Big((3p^2 - 4/3)^{1/2}\Big(\cos((3/2)\epsilon^{-1}p) - \cosh\Big((1/2)\epsilon^{-1}(3p^2 - 4/3)^{1/2}\Big)\Big) + \\
&\quad + i\Big((3p^2 - 4/3)^{1/2}\sin((3/2)\epsilon^{-1}p) - 3p\sinh\Big((1/2)\epsilon^{-1}(3p^2 - 4/3)^{1/2}\Big)\Big)\Big) \\
&\neq 0
\end{aligned}$$

since

$$\cos\Big((3/2)\epsilon^{-1}p\Big) - \cosh\Big((1/2)\epsilon^{-1}(3p^2 - 4/3)^{1/2}\Big) \neq 0 \text{ for every } \epsilon > 0.$$

For $\nu = \pm(2/27)i$ we have by analytic extension:

$$H(\epsilon, \pm(2/27)i) = \exp(\mp(1/3)i\epsilon^{-1})\left(1 \pm i\epsilon^{-1} - \exp(\pm i\epsilon^{-1})\right) \neq 0$$

for every $\epsilon > 0$. □

The $\rho_j(\nu)(1 \leq j \leq 3)$ may be chosen as analytic functions on $V := \{\nu \in \mathbf{C} \,|\, \mathrm{Re}\,\nu \geq 0\} \setminus \{-(2/27)i, (2/27)i\}$ satisfying the conditions:

(i) $\mathrm{Re}\ \rho_1(\nu) < 0 < \mathrm{Re}\ \rho_2(\nu) < \mathrm{Re}\ \rho_3(\nu), \quad \mathrm{Re}\ \nu > 0, \mathrm{Im}\ \nu > 0$

(ii) $\mathrm{Re}\ \rho_1(\nu) < 0 < \mathrm{Re}\ \rho_2(\nu) = \mathrm{Re}\ \rho_3(\nu), \quad \mathrm{Re}\ \nu > 0, \mathrm{Im}\ \nu = 0$

(iii) $\mathrm{Re}\ \rho_1(\nu) < 0 < \mathrm{Re}\ \rho_3(\nu) < \mathrm{Re}\ \rho_2(\nu), \quad \mathrm{Re}\ \nu > 0, \mathrm{Im}\ \nu < 0$

(iv) $$\lim_{\substack{\nu \to -(2/27)i \\ \nu \in V}} \rho_1(\nu) = \lim_{\substack{\nu \to -(2/27)i \\ \nu \in V}} \rho_2(\nu) = (1/3)i,$$
$$\lim_{\substack{\nu \to -(2/27)i \\ \nu \in V}} \rho_3(\nu) = -(2/3)i$$

(v) $$\lim_{\substack{\nu \to (2/27)i \\ \nu \in V}} \rho_1(\nu) = \lim_{\substack{\nu \to (2/27)i \\ \nu \in V}} \rho_3(\nu) = -(1/3)i,$$
$$\lim_{\substack{\nu \to (2/27)i \\ \nu \in V}} \rho_2(\nu) = (2/3)i.$$

Since

$$|\,(\rho_1(\nu) - \rho_2(\nu))(\rho_1(\nu) - \rho_3(\nu))^{-1}\,| \leq 1$$

for $\nu \in \{z \in \mathbf{C} \,|\, \mathrm{Re}\ z \geq 0, \mathrm{Im}\ z = 0\} \cup \{z \in \mathbf{C} \,|\, \mathrm{Re}\ z = 0, \mathrm{Im}\ z \leq 0\}$ and

$$\lim_{\substack{\nu \to \infty \\ \mathrm{Re}\ \nu \geq 0, \mathrm{Im}\ \nu \leq 0}} |\,(\rho_1(\nu) - \rho_2(\nu))(\rho_1(\nu) - \rho_3(\nu))^{-1}\,| = 1$$

we have as a consequence of the maximum principle for analytic functions:

$$|\,(\rho_1(\nu) - \rho_2(\nu))(\rho_1(\nu) - \rho_3(\nu))^{-1}\,| \leq 1$$

for $\nu \in \{z \in \mathbf{C} \,|\, \mathrm{Re}\ z \geq 0, \mathrm{Im}\ z \leq 0\}$. We also have:

$$\left\{(\rho_3(\nu) - \rho_2(\nu))(\rho_1(\nu) - \rho_3(\nu))^{-1} \,|\, \mathrm{Re}\ \nu \geq 0, \mathrm{Im}\ \nu \leq 0\right\} \subset \mathbf{C}\backslash[0, \infty).$$

Indeed, assume $(\rho_3(\nu)-\rho_2(\nu))(\rho_1(\nu)-\rho_3(\nu))^{-1}=t\geq 0$. Using the relation:

$$\sum_{i=1}^{3}\rho_i(\nu)=0,$$

we find:

$$\rho_2(\nu)=-(2t+1)(t+2)^{-1}\rho_1(\nu)$$

$$\rho_3(\nu)=(t-1)(t+2)^{-1}\rho_1(\nu).$$

Thus

$$\begin{aligned}1/3 &= \rho_1(\nu)\rho_2(\nu)+\rho_2(\nu)\rho_3(\nu)+\rho_3(\nu)\rho_1(\nu)=\\ &= -3(t^2+t+1)(t+2)^{-2}\rho_1^2(\nu).\end{aligned}$$

Therefore $\mathrm{Re}\,\rho_1(\nu)=\mathrm{Re}\,\rho_2(\nu)=\mathrm{Re}\,\rho_3(\nu)=0$. Hence $\nu=i\alpha$ with $-2/27\leq\alpha\leq 0$. But for such ν we have: $-i(\rho_3(\nu)-\rho_2(\nu))<0$ and $-i(\rho_1(\nu)-\rho_3(\nu))>0$, so that $(\rho_3(\nu)-\rho_2(\nu))(\rho_1(\nu)-\rho_3(\nu))^{-1}<0$, which contradicts the assumption. Thus

$$\mu\to\mathrm{Im}\left(\log(\rho_3(\nu)-\rho_2(\nu))(\rho_1(\nu)-\rho_3(\nu))^{-1}\right)$$

is bounded on the set $\{z\in\mathbf{C}\,|\,\mathrm{Re}\,z\geq 0,\mathrm{Im}\,z\leq 0\}$.
Let $\mu:(0,\epsilon_0]\to\mathbf{C}$ be such that $\Phi(\epsilon,\mu(\epsilon))=0,\quad\forall\epsilon\in(0,\epsilon_0]$. Setting $\nu(\epsilon):=\epsilon^2\mu(\epsilon)-(2/27)i$, we have: $H(\epsilon,\nu(\epsilon))=0,\quad\forall\epsilon\in(0,\epsilon_0]$. As a consequence of Lemma 4.7 we have: $\mathrm{Re}\,\nu(\epsilon)\geq 0\;\;\forall\epsilon\in(0,\epsilon_0]$. Setting

$$\tilde{\nu}(\epsilon):=\begin{cases}\nu(\epsilon)\text{ if Im }\nu(\epsilon)\leq 0\\ \overline{\nu(\epsilon)}\text{ if Im }\nu(\epsilon)>0\end{cases}$$

we have, since $H(\epsilon,\overline{\nu})=\overline{H(\epsilon,\nu)}$ for $\epsilon>0$:

$$H(\epsilon,\tilde{\nu}(\epsilon))=0,\quad\forall\epsilon\in(0,\epsilon_0]$$

and

$$\tilde{\nu}(\epsilon)\in\{\mu\in\mathbf{C}\,|\,\mathrm{Re}\,\mu\geq 0,\mathrm{Im}\,\mu\leq 0\}\setminus\{-2/27i\},\quad\forall\epsilon\in(0,\epsilon_0].$$

Therefore

$$\begin{aligned}&(\rho_3(\tilde{\nu}(\epsilon))-\rho_2(\tilde{\nu}(\epsilon)))\exp(\epsilon^{-1}\rho_1(\tilde{\nu}(\epsilon)))+\\ &+(\rho_1(\tilde{\nu}(\epsilon))-\rho_3(\tilde{\nu}(\epsilon)))\exp(\epsilon^{-1}\rho_2(\tilde{\nu}(\epsilon)))+\\ &+(\rho_2(\tilde{\nu}(\epsilon))-\rho_1(\tilde{\nu}(\epsilon)))\exp(\epsilon^{-1}\rho_3(\tilde{\nu}(\epsilon)))=0,\quad\forall\epsilon\in(0,\epsilon_0]\end{aligned}$$

implies that

$$\begin{aligned}
&\exp\left(\epsilon^{-1}\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_2\left(\tilde{\nu}(\epsilon)\right)\right)+\right.\\
&\qquad\left.+\log\left(\left(\rho_3\left(\tilde{\nu}(\epsilon)\right)-\rho_2\left(\tilde{\nu}(\epsilon)\right)\right)\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_3\left(\tilde{\nu}(\epsilon)\right)\right)^{-1}\right)\right)=\\
&=-1+\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_2\left(\tilde{\nu}(\epsilon)\right)\right)\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_3\left(\tilde{\nu}(\epsilon)\right)\right)^{-1}\cdot\\
&\qquad\cdot\exp\left(\epsilon^{-1}\left(\rho_3\left(\tilde{\nu}(\epsilon)\right)-\rho_2\left(\tilde{\nu}(\epsilon)\right)\right)\right)\\
&\in\{z\in\mathbf{C}\,|\,|z+1|\le 1, z\neq 0\},\quad\forall\epsilon\in(0,\epsilon_0],
\end{aligned}$$

since

$$\left|\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_2\left(\tilde{\nu}(\epsilon)\right)\right)\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_3\left(\tilde{\nu}(\epsilon)\right)\right)^{-1}\right|\le 1$$

and

$$\operatorname{Re}\left(\rho_3\left(\tilde{\nu}(\epsilon)\right)-\rho_2\left(\tilde{\nu}(\epsilon)\right)\right)\le 0.$$

Therefore the function

$$\begin{aligned}
\epsilon\to\ &\operatorname{Im}\left(\epsilon^{-1}\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_2\left(\tilde{\nu}(\epsilon)\right)\right)+\right.\\
&\qquad\left.+\log\left(\left(\rho_3\left(\tilde{\nu}(\epsilon)\right)-\rho_2\left(\tilde{\nu}(\epsilon)\right)\right)\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_3\left(\tilde{\nu}(\epsilon)\right)\right)^{-1}\right)\right)
\end{aligned}$$

is bounded for $\epsilon\in(0,\epsilon_0]$. Hence, $\operatorname{Im}\left(\epsilon^{-1}\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_2\left(\tilde{\nu}(\epsilon)\right)\right)\right)$ is bounded for $\epsilon\in(0,\epsilon_0]$, too. Therefore

$$\operatorname{Im}\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_2\left(\tilde{\nu}(\epsilon)\right)\right)=O(\epsilon)\text{ for }\epsilon\downarrow 0. \tag{4.41}$$

We also have:

$$\begin{aligned}
&\exp\left(\epsilon^{-1}\left(\rho_2\left(\tilde{\nu}(\epsilon)\right)-\rho_3\left(\tilde{\nu}(\epsilon)\right)\right)\right)=\\
&\qquad\left(\rho_2\left(\tilde{\nu}(\epsilon)\right)-\rho_3\left(\tilde{\nu}(\epsilon)\right)\right)\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_3\left(\tilde{\nu}(\epsilon)\right)\right)^{-1}\cdot\\
&\qquad\qquad\cdot\exp\left(\epsilon^{-1}\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_3\left(\tilde{\nu}(\epsilon)\right)\right)\right)+\\
&\qquad+\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_2\left(\tilde{\nu}(\epsilon)\right)\right)\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)-\rho_3\left(\tilde{\nu}(\epsilon)\right)\right)^{-1}.
\end{aligned}$$

Since the functions

$$\nu\to\left(\rho_2(\nu)-\rho_3(\nu)\right)\left(\rho_1(\nu)-\rho_3(\nu)\right)^{-1}$$

$$\nu\to\left(\rho_1(\nu)-\rho_2(\nu)\right)\left(\rho_1(\nu)-\rho_3(\nu)\right)^{-1}$$

are bounded in $\{\nu\in\mathbf{C}\,|\,\operatorname{Re}\nu\ge 0,\operatorname{Im}\nu\le 0\}$ and $\operatorname{Re}\rho_1(\nu)\le\operatorname{Re}\rho_3(\nu)$ if $\operatorname{Re}\nu\ge 0$ and $\operatorname{Im}\nu\le 0$, we have:

$$\exp\left(\epsilon^{-1}\left(\rho_2\left(\tilde{\nu}(\epsilon)\right)-\rho_3\left(\tilde{\nu}(\epsilon)\right)\right)\right)\text{ is bounded for }\epsilon\in(0,\epsilon_0].$$

Thus

(4.42) $$\operatorname{Re}\left(\rho_2\left(\tilde{\nu}(\epsilon)\right)-\rho_3\left(\tilde{\nu}(\epsilon)\right)\right)=O(\epsilon) \text{ for } \epsilon\downarrow 0.$$

Using (4.41) and (4.42) we get: $\lim_{\epsilon\downarrow 0}\tilde{\nu}(\epsilon)=-(2/27)i$. Therefore $\tilde{\nu}(\epsilon)=\nu(\epsilon)$ or $\tilde{\nu}(\epsilon)=\overline{\nu(\epsilon)}$ for ϵ sufficiently small. Writing

$$\tilde{\nu}(\epsilon)=-(2/27)i+R(\epsilon),\ \lim_{\epsilon\downarrow 0}R(\epsilon)=0$$

and using the equation

$$\left(\rho_1\left(\tilde{\nu}(\epsilon)\right)\right)^3+(1/3)\rho_1\left(\tilde{\nu}(\epsilon)\right)+\tilde{\nu}(\epsilon)=0,$$

we find:

$$\begin{aligned}
\rho_1\left(\tilde{\nu}(\epsilon)\right)&=(1/3)i-\exp(i\pi/4)\left(R(\epsilon)\right)^{1/2}+O\left(R(\epsilon)\right)\\
\rho_2\left(\tilde{\nu}(\epsilon)\right)&=(1/3)i+\exp(i\pi/4)\left(R(\epsilon)\right)^{1/2}+O\left(R(\epsilon)\right)\\
\rho_3\left(\tilde{\nu}(\epsilon)\right)&=-(2/3)i+O\left(R(\epsilon)\right)
\end{aligned}$$

as $\epsilon\downarrow 0$. Using (4.41) and (4.42), we get:

$$\begin{aligned}
\operatorname{Im}\left(\exp(i\pi/4)\left(R(\epsilon)\right)^{1/2}\right)&=O(\epsilon)+O\left(R(\epsilon)\right)\\
\operatorname{Re}\left(\exp(i\pi/4)\left(R(\epsilon)\right)^{1/2}\right)&=O(\epsilon)+O\left(R(\epsilon)\right)
\end{aligned}$$

as $\epsilon\downarrow 0$. Hence one finds:

$$\exp(i\pi/4)\left(R(\epsilon)\right)^{1/2}=O(\epsilon)+O\left(R(\epsilon)\right) \text{ as } \epsilon\downarrow 0.$$

Therefore we have $(R(\epsilon))^{1/2}=O(\epsilon)$ as $\epsilon\downarrow 0$. Thus $R(\epsilon)=O(\epsilon^2)$ as $\epsilon\downarrow 0$, and $\tilde{\mu}(\epsilon):=\epsilon^{-2}R(\epsilon)=\epsilon^{-2}(\tilde{\nu}(\epsilon)+(2/27)i)$ is bounded for $\epsilon\in(0,\epsilon_0]$. Moreover: $\Phi\left(\epsilon,\tilde{\mu}(\epsilon)\right)=0, \forall\epsilon\in(0,\epsilon_0]$. Using that $\epsilon^{-1}\Phi(\epsilon,\mu)=i^{-1/2}\mu^{-1/2}\sin(i^{-1/2}\mu^{1/2})+O(\epsilon)$ as $\epsilon\downarrow 0$, where $O(\epsilon)$ is uniform on the compact subsets of $\mathbf{C}$, and Rouché's theorem we reach the conclusion that there is a $k\in\mathbf{N}^*$ such that $\tilde{\mu}(\epsilon)=\mu_k^\epsilon$ for ϵ sufficiently small. If $\tilde{\nu}(\epsilon)=\nu(\epsilon)$ for ϵ sufficiently small we have:

$$\mu(\epsilon)=\epsilon^{-2}(\nu(\epsilon)+(2/27)i)=\epsilon^{-2}(\tilde{\nu}(\epsilon)+(2/27)i)=\tilde{\mu}(\epsilon)=\mu_k^\epsilon.$$

If $\tilde{\nu}(\epsilon)=\overline{\nu(\epsilon)}$ for ϵ sufficiently small we find:

$$\begin{aligned}
\mu(\epsilon)&=\epsilon^{-2}(\nu(\epsilon)+(2/27)i)=\epsilon^{-2}(\overline{\tilde{\nu}(\epsilon)}+(2/27)i)=\\
&=\epsilon^{-2}(\overline{-(2/27)i+\epsilon^2\tilde{\mu}(\epsilon)}+(2/27)i)=\epsilon^{-2}\left((4/27)i+\epsilon^2\overline{\mu_k^\epsilon}\right)=\tau_k^\epsilon.
\end{aligned}$$

Summarizing we have proved the following theorem:

Theorem 4.3 *Let μ_k^ϵ and τ_k^ϵ, defined by (4.36), be the eigenvalues of eigenvalue problem (4.25), (4.24). Further let $\mu(\epsilon), \epsilon \in (0, \epsilon_0]$, be a continuous function satisfying the equation $\Phi(\epsilon, \mu(\epsilon)) = 0, \forall \epsilon \in (0, \epsilon_0]$. Then there is a $k \in \mathbf{N}^*$ such that either $\mu(\epsilon) = \mu_k^\epsilon$ or $\mu(\epsilon) = \tau_k^\epsilon$, provided that ϵ is sufficiently small.*

Now we consider the following question: for which $\epsilon > 0$ does problem (4.25), (4.24) have purely imaginary eigenvalues? It turns out that for every $\epsilon > 0$ problem (4.25), (4.24) only has a finite number of purely imaginary eigenvalues (for most ϵ this number will be zero) but there is a sequence $\epsilon_1 > \epsilon_2 > \ldots > 0$, $\epsilon_p \downarrow 0$ as $p \to \infty$ such that the number of purely imaginary eigenvalues of the corresponding eigenvalue problems goes to infinity.

For $\nu = i\alpha$ with $-2/27 < \alpha < 2/27$ we may write $\nu = i(p^3 - (1/3)p)$ with $-1/3 < p < 1/3$. The $\rho_i(\nu)\,(1 \le i \le 3)$ may be written in the following form:

$$\rho_1(\nu) = ip \tag{4.43}$$

$$\rho_2(\nu) = -(1/2)ip + (1/2)i\left((4/3) - 3p^2\right)^{1/2} \tag{4.44}$$

$$\rho_3(\nu) = -(1/2)ip - (1/2)i\left((4/3) - 3p^2\right)^{1/2}. \tag{4.45}$$

Substituting (4.43)-(4.45) into (4.37), we get:

$$\begin{aligned}
&H(\epsilon, \nu) = \\
&= i\Big\{-i\left((4/3) - 3p^2\right)^{1/2} \exp(i\epsilon^{-1}p) + \\
&\quad + \left((3/2)ip + (1/2)i\left((4/3) - 3p^2\right)^{1/2}\right) \cdot \\
&\qquad \cdot \exp\left(-(1/2)i\epsilon^{-1}p + (1/2)i\epsilon^{-1}\left((4/3) - 3p^2\right)^{1/2}\right) + \\
&\quad + \left(-(3/2)ip + (1/2)i\left((4/3) - 3p^2\right)^{1/2}\right) \cdot \\
&\quad \cdot \exp\left(-(1/2)i\epsilon^{-1}p - (1/2)i\epsilon^{-1}\left((4/3) - 3p^2\right)^{1/2}\right)\Big\} \cdot \\
&\quad \cdot \left\{\left((4/3) - 3p^2\right)^{1/2}\left((1/3) - 3p^2\right)\right\}^{-1} = \\
&= \left\{\left((4/3) - 3p^2\right)^{1/2}\left((1/3) - 3p^2\right)\right\}^{-1} \exp\left(-(1/2)i\epsilon^{-1}p\right) \cdot \\
&\quad \cdot \Big\{\left((4/3) - 3p^2\right)^{1/2}\left(\cos\left((3/2)\epsilon^{-1}p\right) + \right. \\
&\qquad \left. - \cos\left((1/2)\epsilon^{-1}\left((4/3) - 3p^2\right)^{1/2}\right)\right) +
\end{aligned}$$

$$+i\left(\left((4/3)-3p^2\right)^{1/2}\sin\left((3/2)\epsilon^{-1}p\right)+\right.$$
$$\left.-3p\sin\left((1/2)\epsilon^{-1}\left((4/3)-3p^2\right)^{1/2}\right)\right)\Big\}.$$

Since $\left((4/3)-3p^2\right)^{1/2} > 3|p|$ for every $-1/3 < p < 1/3$, we have

$$H(\epsilon,\nu)=0$$

iff

(4.46) $$(3/2)\epsilon^{-1}p = k\pi,\ (1/2)\epsilon^{-1}\left((4/3)-3p^2\right)^{1/2} = l\pi,$$
$$k\in\mathbf{Z}, l\in\mathbf{N}^*, k\equiv l\,(\mathrm{mod}\ 2).$$

Thus a necessary condition for p in order to have $\nu = i(p^3-(1/3)p)$ as a zero of $H(\epsilon,\nu)$ for some $\epsilon>0$ is the following one:

$$3p\left((4/3)-3p^2\right)^{-1/2}\in\mathbf{Q}\cap(-1,1).$$

Choosing $q\in\mathbf{Q}\cap(-1,1)$, the equation $3p\left((4/3)-3p^2\right)^{-1/2}=q$ is solved by

(4.47) $$p=(2/3)q(3+q^2)^{-1/2}.$$

Writing $q=q_1/q_2, q_1\in\mathbf{Z}, q_2\in\mathbf{N}^*, |q_1|<q_2, (q_1,q_2)=1$, the conditions (4.46) are equivalent with:

(4.48) $$\epsilon=\pi^{-1}\left((nq_1)^2+3(nq_2)^2\right)^{-1/2}, n\in\mathbf{N}^*, nq_1\equiv nq_2\,(\mathrm{mod}\,2).$$

Thus with $q\in\mathbf{Q}\cap(-1,1)$, p given by (4.47), $\nu=i(p^3-(1/3)p)$ the equation $H(\epsilon,\nu)=0$ is solved by every ϵ satisfying (4.48). On the other hand, fixing $\epsilon>0$, we have:

$$H(\epsilon,\nu)=0 \text{ has a purely imaginary solution } \nu$$

iff

the equation $x^2+3y^2=(\pi\epsilon)^{-2}$ has a solution $(x,y)\in\mathbf{Z}\times\mathbf{N}^*$

satisfying $|x|<y$ and $x\equiv y\,(\mathrm{mod}\ 2)$.

In the latter case we have: $H(\epsilon,\nu)=0$ for $\nu=i(p^3-(1/3)p)$ where p is given by (4.47) and $q=x/y$, i.e. $\nu=(2/27)ix(x^2-9y^2)(x^2+3y^2)^{-3/2}$. Therefore

$$\mu=\epsilon^{-2}(\nu+(2/27)i)=(2/27)i\pi^2\left(x(x^2-9y^2)(x^2+3y^2)^{-1/2}+x^2+3y^2\right)$$

is a purely imaginary eigenvalue of eigenvalue problem (4.25), (4.24). Let $\epsilon_1 > \epsilon_2 > \ldots > 0, \epsilon_p \downarrow 0$ as $p \to \infty$, be the sequence of all $\epsilon > 0$ such that the equation

$$x^2 + 3y^2 = (\pi\epsilon)^{-2} \tag{4.49}$$

has a solution $(x, y) \in \mathbf{Z} \times \mathbf{N}^*$ satisfying

$$|x| < y,\ x \equiv y \,(\mathrm{mod}\ 2). \tag{4.50}$$

Let n_p be the number of solutions of (4.49), (4.50) with $\epsilon = \epsilon_p$ and denote these solutions by $(x_{p,k}, y_{p,k}), 1 \leq k \leq n_p$. Then there is a subsequence $\epsilon_{p_1} > \epsilon_{p_2} > \ldots > 0$ such that $\lim_{k\to\infty} n_{p_k} = +\infty$. Now we have proved the following theorem:

Theorem 4.4 *Let $\epsilon_p, n_p, x_{p,k}, y_{p,k}, 1 \leq k \leq n_p$, be defined as hereabove. Then for each $p \in \mathbf{N}^*$ eigenvalue problem (4.25), (4.24) with $\epsilon = \epsilon_p$ precisely has n_p purely imaginary eigenvalues $\mu_k(\epsilon_p), 1 \leq k \leq n_p$, given by the formula:*

$$\begin{aligned}\mu_k(\epsilon_p) = \ & (2/27)i\pi^2\left(x_{p,k}(x_{p,k}^2 - 9y_{p,k}^2)(x_{p,k}^2 + 3y_{p,k}^2)^{-1/2} + x_{p,k}^2 + 3y_{p,k}^2\right), \\ & 1 \leq k \leq n_p.\end{aligned}$$

Moreover there is a subsequence $\epsilon_{p_1} > \epsilon_{p_2} > \ldots > 0, \epsilon_{p_k} \downarrow 0$ as $k \to \infty$, such that $\lim_{k\to\infty} n_{p_k} = +\infty$.

Chapter 5

Remark on finite difference approximations

5.1 The Cauchy problem

Let $\mathbf{R}_h = h\mathbf{Z} \subset \mathbf{R}$ be the grid (lattice) with mesh-size h and let Θ_h be the shift operator on $\mathbf{R}_h$ to the left, Θ_h^{-1} being the shift to the right. Consider the following finite difference approximation in the space x_1-variable of the operator $\partial_{x_0} + \partial_{x_1}$:

$$(5.1)\qquad L_h(\partial_{x_0}, \Theta_h) := \partial_{x_0} + (2h)^{-1}(\Theta_h - \Theta_h^{-1}), \quad x_0 \in \mathbf{R}, x_1 \in \mathbf{R}_h.$$

The fundamental solution $E_h(x_0, x_1)$ of the Cauchy problem for L_h is given by the following expression:

$$(5.2)\qquad E_h(x_0, x_1) = (2\pi h)^{-1} \int_{-\pi}^{\pi} \exp\left(ih^{-1}\left(-x_0 \sin u + x_1 u\right)\right) du.$$

If $|x_1| < |x_0|$ the asymptotic expansion of E_h is found, using the stationary-phase method:

$$(5.3)\quad E_h(x_0, x_1) \sim 2\mathrm{Re}\left(\exp\left(ih^{-1}\phi(x_0, x_1)\right) \sum_{n=0}^{\infty} a_n(x_0, x_1) h^{n-1/2}\right),$$

where

$$\begin{aligned} a_0(x_0, x_1) &= 2^{-1/2} \cdot \pi^{-1/2} \cdot \left(x_0^2 - x_1^2\right)^{-1/4} \cdot \exp(\mathrm{sgn}(x_0) i\pi/4), \\ \phi(x_0, x_1) &= -\mathrm{sgn}(x_0)\left(x_0^2 - x_1^2\right)^{1/2} + x_1 \arccos(x_1/x_0). \end{aligned}$$

E_h is found to be exponentially small as $h \downarrow 0$ for $|x_1| > |x_0|$. Therefore

$$(5.4)\qquad \text{sing supp } E_h = \{(x_0, x_1) \in \mathbf{R}^2 \,|\, |x_1| \le |x_0|\}.$$

Summarizing we have:

Theorem 5.1 *The fundamental solution E_h of the Cauchy problem for L_h, given by* (5.1), *is given by* (5.2), *its asymptotic expansion by* (5.3) *and its singular support by* (5.4).

5.2 The half-line

Let $\mathbf{R}_{h,+} = \mathbf{R}_+ \cap \mathbf{R}_h$ and $\mathbf{R}_{h,1}^- = \mathbf{R}_1^- \cap \mathbf{R}_h$ where $\mathbf{R}_1^- = \{x \in \mathbf{R} \,|\, x < 1\}$. Consider the following mixed problems for the operator L_h:

$$(5.5) \qquad \begin{cases} L_h(\partial_{x_0}, \Theta_h) u_h^+(x_0, x_1) = 0, & (x_0, x_1) \in \mathbf{R}_+ \times \mathbf{R}_{h,+}, \\ u_h^+(0, x_1) = 0, & x_1 \in \mathbf{R}_{h,+}, \\ u_h^+(x_0, 0) = \psi_0(x_0), & x_0 \in \mathbf{R}_+, \end{cases}$$

and

$$(5.6) \qquad \begin{cases} L_h(\partial_{x_0}, \Theta_h) u_h^-(x_0, x_1) = 0, & (x_0, x_1) \in \mathbf{R}_+ \times \mathbf{R}_{h,1}^-, \\ u_h^-(0, x_1) = 0, & x_1 \in \mathbf{R}_{h,1}^-, \\ u_h^-(x_0, 1) = \psi_1(x_0), & x_0 \in \mathbf{R}_+. \end{cases}$$

We seek the solutions u_h^+ and u_h^- of (5.5) and (5.6), which vanish for $x_0 < 0$. Let $P_h^+(x_0, x_1)$ and $P_h^-(x_0, x_1)$ be the kernels of the operators, solving these problems, i.e.

$$u_h^+(x_0, x_1) = \int_0^{x_0} P_h^+(x_0 - y_0, x_1) \psi_0(y_0) dy_0, \qquad (x_0, x_1) \in \mathbf{R}_+ \times \mathbf{R}_{h,+},$$
$$u_h^-(x_0, x_1) = \int_0^{x_0} P_h^-(x_0 - y_0, x_1) \psi_1(y_0) dy_0, \qquad (x_0, x_1) \in \mathbf{R}_+ \times \mathbf{R}_{h,1}^-.$$

Using the Fourier transform $F_{x_0 \to \xi_0}$ one finds:

$$(5.7) \quad P_h^+(x_0, x_1) = (2\pi h)^{-1} \int_{\mathbf{R}} \exp\left(h^{-1}\left(ix_0\xi_0 + x_1 \log \Theta_+(\xi_0)\right)\right) d\xi_0$$

and

$$(5.8) P_h^-(x_0, x_1) = (2\pi h)^{-1} \int_{\mathbf{R}} \exp\left(h^{-1}\left(ix_0\xi_0 + (x_1 - 1) \log \Theta_-(\xi_0)\right)\right) d\xi_0,$$

where

$$\Theta_\pm(\xi) = -i\xi \pm \chi_{(-1,1)}(\xi)(1 - \xi^2)^{1/2} \pm i \operatorname{sgn}(\xi)(1 - \chi_{(-1,1)}(\xi))(\xi^2 - 1)^{1/2}.$$

Using the stationary-phase method for P_h^+ one finds, for $x_0 > x_1$, as $h \downarrow 0$:

$$(5.9) \quad P_h^+(x_0, x_1) \sim 2\mathrm{Re}\left(\exp(ih^{-1}\phi_+(x_0, x_1)) \sum_{n=0}^{\infty} b_n(x_0, x_1) h^{n-1/2}\right),$$

where

$$\begin{aligned} b_0(x_0,x_1) &= 2^{-1/2}\cdot\pi^{-1/2}\cdot\exp(-i\pi/4)\cdot x_1\cdot x_0^{-1}\cdot(x_0^2-x_1^2)^{-1/4},\\ \phi_+(x_0,x_1) &= (x_0^2-x_1^2)^{1/2}-x_1\arcsin\left(1-x_1^2x_0^{-2}\right)^{1/2}. \end{aligned}$$

$P_h^+(x_0,x_1)$ is exponentially small as $h\downarrow 0$, for $x_0<x_1$.

Similarly one finds the asymptotic expansion of $P_h^-(x_0,x_1)$ as $h\downarrow 0$, for $1-x_1<x_0$:

$$(5.10)\quad P_h^-(x_0,x_1)\sim 2\mathrm{Re}\,(\exp(ih^{-1}\phi_-(x_0,x_1))\sum_{n=0}^{\infty}c_n(x_0,x_1)h^{n-1/2}),$$

where

$$\begin{aligned} &c_0(x_0,x_1)=2^{-1/2}\cdot\pi^{-1/2}\cdot\exp(-i\pi/4)\cdot(x_0^2-(1-x_1)^2)^{-1/4}\cdot(1-x_1)\cdot x_0^{-1},\\ &\phi_-(x_0,x_1)=(x_0^2-(1-x_1)^2)^{1/2}+(1-x_1)\arccos((x_1-1)x_0^{-1}), \end{aligned}$$

whereas $P_h^-(x_0,x_1)$ is exponentially small as $h\downarrow 0$ for $x_0<1-x_1$. Thus one has:

$$\begin{aligned} &(5.11)\qquad \text{sing supp } P_h^+ = \{(x_0,x_1)\in\overline{\mathbf{R}_+}\times\overline{\mathbf{R}_+}\,|\,x_0\geq x_1\},\\ &(5.12)\qquad \text{sing supp } P_h^- = \{(x_0,x_1)\in\overline{\mathbf{R}_+}\times\overline{\mathbf{R}_1^-}\,|\,x_0\geq 1-x_1\}. \end{aligned}$$

Summarizing we have:

Theorem 5.2 *The Poisson kernels $P_h^{\pm}$, solving the mixed problems* (5.5), (5.6), *are given by* (5.7), (5.8), *their asymptotic expansions by* (5.9), (5.10) *and their singular supports by* (5.11), (5.12).

5.3 A finite interval

The normalized eigenfunctions ϕ_k^h and eigenvalues λ_k^h ($k\in\mathbf{N}, 0<k<h^{-1}$) of the eigenvalue problem:

$$(5.13)\quad \begin{cases} (2h)^{-1}(\Theta_h-\Theta_h^{-1})u(x)=\lambda u(x), & x\in\mathbf{U}_h=(0,1)\cap\mathbf{R}_h,\\ u(0)=u(1)=0 \end{cases}$$

are

$$(5.14)\qquad \phi_k^h(x)=\sqrt{2}\exp\left(\pi ix(2h)^{-1}\right)\sin(\pi kx),$$

$$(5.15)\qquad \lambda_k^h=ih^{-1}\cos(\pi kh).$$

Therefore the Green kernel G_U^h, corresponding to the following mixed problem on a finite interval:

$$\text{(5.16)} \quad \begin{cases} L_h(\partial_{x_0}, \Theta_h)u_h(x_0, x_1) = f(x_0, x_1), & (x_0, x_1) \in \mathbf{R}_+ \times \mathbf{U}_h, \\ u_h(0, x_1) = \phi(x_1), & x_1 \in \mathbf{U}_h, \\ u_h(x_0, 0) = \psi_0(x_0), & x_0 \in \mathbf{R}_+, \\ u_h(x_0, 1) = \psi_1(x_0), & x_0 \in \mathbf{R}_+ \end{cases}$$

is given by the following expression:

$$G_U^h(x_0, x_1, y_1) = \sum_{0<k<h^{-1}} \exp(-\lambda_k^h x_0)\phi_k^h(x_1)\overline{\phi_k^h(y_1)}.$$

For the solution of the problem:

$$\begin{cases} L_h(\partial_{x_0}, \Theta_h)u_h(x_0, x_1) = 0, & (x_0, x_1) \in \mathbf{R}_+ \times \mathbf{U}_h, \\ u_h(0, x_1) = 0, & x_1 \in \mathbf{U}_h, \\ u_h(x_0, 0) = \psi_0(x_0), & x_0 \in \mathbf{R}_+, \\ u_h(x_0, 1) = 0, & x_0 \in \mathbf{R}_+, \end{cases}$$

with $\psi_0(0) = 0$, we write

$$u_h(x_0, x_1) = (1 - x_1)\psi_0(x_0) + v_h(x_0, x_1)$$

where v_h is the solution of:

$$\begin{cases} L_h(\partial_{x_0}, \Theta_h)v_h(x_0, x_1) = f(x_0, x_1), & (x_0, x_1) \in \mathbf{R}_+ \times \mathbf{U}_h, \\ v_h(0, x_1) = 0, & x_1 \in \mathbf{U}_h, \\ v_h(x_0, 0) = v_h(x_0, 1) = 0, & x_0 \in \mathbf{R}_+, \end{cases}$$

with

$$f(x_0, x_1) = \psi_0(x_0) - (1 - x_1)(\partial_{x_0}\psi_0)(x_0).$$

Therefore

$$\begin{aligned} u_h(x_0, x_1) &= \\ &= (1 - x_1)\psi_0(x_0) + \\ &\quad + \int_0^{x_0} h \sum_{y_1 \in U_h} G_U^h(x_0 - y_0, x_1, y_1) f(y_0, y_1) dy_0 = \\ &= (1 - x_1)\psi_0(x_0) + \\ &\quad + \int_0^{x_0} h \sum_{y_1 \in U_h} \sum_{0<k<h^{-1}} \exp\left(-\lambda_k^h (x_0 - y_0)\right) \phi_k^h(x_1)\overline{\phi_k^h(y_1)} f(y_0, y_1) dy_0 = \\ &= (1 - x_1)\psi_0(x_0) + \\ &\quad - h \sum_{y_1 \in U_h} \sum_{0<k<h^{-1}} \phi_k^h(x_1)\overline{\phi_k^h(y_1)}(1 - y_1)\psi_0(x_0) + \\ &\quad + \int_0^{x_0} h \sum_{y_1 \in U_h} \sum_{0<k<h^{-1}} \exp\left(-\lambda_k^h (x_0 - y_0)\right) \phi_k^h(x_1)\overline{\phi_k^h(y_1)} \cdot \\ &\qquad \cdot (1 + (1 - y_1)\lambda_k^h)\psi_0(y_0) dy_0 = \\ &= \int_0^{x_0} P_{h,U}^+(x_0 - y_0, x_1)\psi_0(y_0) dy_0, \end{aligned}$$

where the Poisson-kernel $P^+_{h,U}$ is given by the following expression:

$$(5.17)\qquad P^+_{h,U}(x_0,x_1) = \\ = h\sum_{y_1\in U_h}\sum_{0<k<h^{-1}} \exp(-\lambda^h_k x_0)\phi^h_k(x_1)\overline{\phi^h_k(y_1)}(1+(1-y_1)\lambda^h_k).$$

Similarly the solution u_h of:

$$\begin{cases} L_h(\partial_{x_0},\Theta_h)u_h(x_0,x_1)=0, & (x_0,x_1)\in \mathbf{R}_+\times \mathbf{U}_h,\\ u_h(0,x_1)=0, & x_1\in \mathbf{U}_h,\\ u_h(x_0,0)=0, & x_0\in\mathbf{R}_+,\\ u_h(x_0,1)=\psi_1(x_0), & x_0\in\mathbf{R}_+, \end{cases}$$

with $\psi_1(0)=0$, is given by:

$$u_h(x_0,x_1)=\int_0^{x_0} P^-_{h,U}(x_0-y_0,x_1)\psi_1(y_0)dy_0,$$

where

$$(5.18)\, P^-_{h,U}(x_0,x_1) = h\sum_{y_1\in U_h}\sum_{0<k<h^{-1}} \exp(-\lambda^h_k x_0)\phi^h_k(x_1)\overline{\phi^h_k(y_1)}(-1+y_1\lambda^h_k).$$

In order to get asymptotic formulae for $P^\pm_{h,U}$ we use the kernels $P^\pm_h$, given by (5.7), (5.8), solving the mixed problems on the corresponding half-lines.

Let $n\in \mathrm{N}$. Denote

$$\begin{aligned} \alpha_h(\xi_0,x_1) &:= F_{x_0\to\xi_0}(P^+_h(x_0,x_1))\\ \beta_h(\xi_0,x_1) &:= F_{x_0\to\xi_0}(P^-_h(x_0,x_1)), \end{aligned}$$

and let

$$\begin{aligned} R^n_h(x_0,x_1) &:= F^{-1}_{\xi_0\to x_0}\Big(\ \alpha_h(\xi_0,x_1)\sum_{k=0}^{n}(\alpha_h(\xi_0,1)\beta_h(\xi_0,0))^k + \\ &\qquad -\beta_h(\xi_0,x_1)\sum_{k=0}^{n}(\alpha_h(\xi_0,1))^{k+1}(\beta_h(\xi_0,0))^k\Big),\\ S^n_h(x_0,x_1) &:= F^{-1}_{\xi_0\to x_0}\Big(\ \beta_h(\xi_0,x_1)\sum_{k=0}^{n}(\alpha_h(\xi_0,1)\beta_h(\xi_0,0))^k + \\ &\qquad -\alpha_h(\xi_0,x_1)\sum_{k=0}^{n}(\alpha_h(\xi_0,1))^{k}(\beta_h(\xi_0,0))^{k+1}\Big). \end{aligned}$$

Now one easily checks that R^n_h and S^n_h satisfy

$$L_h(\partial_{x_0},\Theta_h)u(x_0,x_1)=0$$

in $\mathbf{R} \times \mathbf{U}_h$. Moreover R_h^n, S_h^n vanish for $x_0 < 0$ and

$$\begin{aligned} R_h^n(x_0, 0) &= \delta(x_0) + r_h^n(x_0), \\ R_h^n(x_0, 1) &= 0, \\ S_h^n(x_0, 0) &= 0, \\ S_h^n(x_0, 1) &= \delta(x_0) + s_h^n(x_0), \end{aligned}$$

where

$$\begin{aligned} r_h^n(x_0) = s_h^n(x_0) &= -F^{-1}_{\xi_0 \to x_0}\left((\alpha_h(\xi_0,1))^{n+1}(\beta_h(\xi_0,0))^{n+1}\right) = \\ &= -(2\pi h)^{-1}\int_{\mathbf{R}} \exp\ \left(h^{-1}\left(i\xi_0 x_0 + (n+1)\log\left(\Theta_+\left(\xi_0\right)\right) + \right.\right. \\ &\qquad \left.\left. -(n+1)\log\left(\Theta_-\left(\xi_0\right)\right)\right)\right) d\xi_0. \end{aligned}$$

Using the saddle-point method one finds that $r_h^n = s_h^n$ is exponentially small as $h \downarrow 0$ for $x_0 < 2n+2$. Therefore, $P^+_{h,U}$ has the same asymptotic expansion as R_h^n and $P^-_{h,U}$ has the same asymptotic expansion as S_h^n in the region $\{(x_0, x_1) \in \mathbf{R}_+ \times \mathbf{U}_h |\, x_0 < 2n+2\}$. Now one gets the asymptotic expansions of $P^{\pm}_{h,U}$, again using the stationary-phase method:

(5.19) $\quad P^+_{h,U}(x_0, x_1) \sim$

$$\sim \begin{cases} \displaystyle\sum_{k=0}^{n} 2\mathrm{Re}\left(\exp\left(ih^{-1}\phi_k^+(x_0,x_1)\right)\cdot\sum_{j=0}^{\infty} b_j\left(x_0, x_1+2k\right)h^{j-1/2}\right) + \\ \displaystyle -\sum_{k=0}^{n-1} 2\mathrm{Re}\left(\exp\left(ih^{-1}\psi_k^+(x_0,x_1)\right)\cdot\sum_{j=0}^{\infty} c_j\left(x_0, x_1-2k-1\right)h^{j-1/2}\right) \\ \text{for } x_0 + x_1 - 2 < 2n < x_0 - x_1, \\ \displaystyle\sum_{k=0}^{n-1} 2\mathrm{Re}\left(\exp\left(ih^{-1}\phi_k^+(x_0,x_1)\right)\cdot\sum_{j=0}^{\infty} b_j\left(x_0, x_1+2k\right)h^{j-1/2}\right) + \\ \displaystyle -\sum_{k=0}^{n-1} 2\mathrm{Re}\left(\exp\left(ih^{-1}\psi_k^+(x_0,x_1)\right)\cdot\sum_{j=0}^{\infty} c_j\left(x_0, x_1-2k-1\right)h^{j-1/2}\right) \\ \text{for } x_0 - x_1 < 2n < x_0 + x_1, \end{cases}$$

where

$$\phi_k^+(x_0, x_1) = \phi^+(x_0, x_1 + 2k) - k\pi$$

$$\psi_k^+(x_0, x_1) = \phi^-(x_0, x_1 - (2k+1)) - (k+1)\pi$$

$P^+_{h,U}$ is exponentially small as $h \downarrow 0$ for $x_0 < x_1$.

(5.20) $P^-_{h,U}(x_0, x_1) \sim$

$$\sim \begin{cases} \sum_{k=0}^{n} 2\mathrm{Re}\left(\exp\left(ih^{-1}\phi_k^-(x_0,x_1)\right)\sum_{j=0}^{\infty} c_j\,(x_0, x_1 - 2k)\,h^{j-1/2}\right) + \\ -\sum_{k=0}^{n-1} 2\mathrm{Re}\left(\exp\left(ih^{-1}\psi_k^-(x_0,x_1)\right)\sum_{j=0}^{\infty} b_j\,(x_0, x_1 + 2k + 1)\,h^{j-1/2}\right) \\ \text{for } x_0 + x_1 > 2n+1 > x_0 - x_1, \\ \sum_{k=0}^{n} 2\mathrm{Re}\left(\exp\left(ih^{-1}\phi_k^-(x_0,x_1)\right)\sum_{j=0}^{\infty} c_j\,(x_0, x_1 - 2k)\,h^{j-1/2}\right) + \\ -\sum_{k=0}^{n} 2\mathrm{Re}\left(\exp\left(ih^{-1}\psi_k^-(x_0,x_1)\right)\cdot\sum_{j=0}^{\infty} b_j\,(x_0, x_1 + 2k + 1)\,h^{j-1/2}\right) \\ \text{for } x_0 - x_1 > 2n+1 > x_0 + x_1 - 2, \end{cases}$$

where

$$\phi_k^-(x_0,x_1) = \phi^-(x_0, x_1 - 2k) - k\pi$$

$$\psi_k^-(x_0,x_1) = \phi^+(x_0, x_1 + 2k + 1) - (k+1)\pi.$$

$P^-_{h,U}$ is exponentially small as $h \downarrow 0$ for $x_0 + x_1 < 1$. The functions b_j, c_j, $j \in \mathbf{N}$, are the same as in the asymptotic expansions of $P^\pm_{h,U}$, given by (5.9), (5.10). Thus we have:

(5.21) $$\text{sing supp } P^+_{h,U} = \{(x_0,x_1) \in \overline{\mathbf{R}_+} \times \overline{U} \mid x_0 \geq x_1\}$$

(5.22) $$\text{sing supp } P^-_{h,U} = \{(x_0,x_1) \in \overline{\mathbf{R}_+} \times \overline{U} \mid x_0 + x_1 \geq 1\}.$$

Writing for the Green function:

(5.23) $$\begin{aligned} G_U^h(x_0, x_1, y_1) = \; & E_h(x_0, x_1 - y_1) + \\ & -\int_0^{x_0} P^+_{h,U}(x_0 - y_0, x_1) E_h(y_0, -y_1)\,dy_0 + \\ & -\int_0^{x_0} P^-_{h,U}(x_0 - y_0, x_1) E_h(y_0, 1 - y_1)\,dy_0 \end{aligned}$$

one gets

(5.24) $$\text{sing supp } G_U^h = \{(x_0,x_1) \in \overline{\mathbf{R}_+} \times \overline{U} \mid |x_1 - y_1| \leq x_0\}.$$

Summarizing we have:

Theorem 5.3 *All the eigenvalues of eigenvalue problem* (5.13) *are purely imaginary and are given by* (5.15), *the associated eigenfunctions by* (5.14). *The Poisson kernels $P^{\pm}_{h,U}$, solving the corresponding mixed problem* (5.16), *are given by* (5.17), (5.18), *their asymptotic expansions by* (5.19), (5.20) *and their singular supports by* (5.21), (5.22). *The Green kernel G^h_U is given by* (5.23) *and its singular support by* (5.24).

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Samenvatting

De Korteweg-de Vries vergelijking is een niet-lineaire partiële differentiaalvergelijking die een wiskundig model vormt voor de beschrijving van de voortplanting van golven in water. In dit proefschrift worden linearisaties van de Korteweg-de Vries vergelijking bekeken, waarbij de dispersieve term klein wordt verondersteld. Voor de bijbehorende singulier gestoorde lineaire partiële differentiaalvergelijkingen worden in detail het Cauchy probleem en gemengde begin-randwaarde problemen op een halfrechte en een eindig interval bekeken. De singulariteiten van de familie kernen, die deze problemen oplossen, worden onderzocht als $\epsilon \to 0$, waarbij de parameter $\epsilon > 0$ de mate van dispersie karakteriseert. Speciale aandacht wordt besteed aan het eigenwaarde probleem voor het stationaire gedeelte van de differentiaaloperator op een eindig interval. Het blijkt dat de bijbehorende tijdsafhankelijke differentiaaloperator, die dispersief is in geval van het Cauchy probleem, voor sommige termen in feite dissipatief wordt, dat wil zeggen gekarakteriseerd wordt door verlies van energie, terwijl voor andere dispersieve termen de dispersieve aard behouden blijft.

In concreto worden in dit proefschrift de singulier gestoorde partiële differentiaaloperatoren $P^{\pm}(\epsilon, \partial_{x_0}, \partial_{x_1}) := \partial_{x_0} + \partial_{x_1} \pm \epsilon^2 \partial_{x_1}^3$ bekeken, waarbij $\epsilon > 0$ een kleine parameter is, x_0 de tijdvariabele en x_1 de ruimtevariabele. In hoofdstuk 1 wordt het Cauchy probleem voor P^- onderzocht en wordt de singuliere drager van de fundamentele oplossing bepaald. Ook worden a posteriori afschattingen voor de oplossingen van het Cauchy probleem verkregen in termen van Sobolev-normen. De resultaten voor P^+ zijn analoog. In hoofdstuk 2 wordt voor $x_0 > 0$ een begin-randwaarde probleem op de halfrechte $\{x_1 \in \mathbf{R} \mid x_1 > 0\}$ bekeken voor P^- met coërcieve randvoorwaarden. Het begin-randwaarde probleem voor P^+ is voor $x_0 > 0$ niet goed gesteld. In hoofdstuk 3 wordt voor $x_0 > 0$ een begin-randwaarde probleem op een eindig interval bekeken voor P^-. In hoofdstuk 4 wordt het eigenwaarde probleem voor de operatoren $Q^{\pm}(\epsilon, \partial_{x_1}) := \pm\partial_{x_1} - \epsilon^2 \partial_{x_1}^3$ bekeken. Voor Q^+ vinden we dat, mits ϵ voldoende klein is, alle eigenwaarden reëel en positief zijn. Bovendien divergeren deze eigenwaarden naar $+\infty$ als $\epsilon \to 0$. Daarom is de bijbehorende differentiaaloperator P^- dissipatief. Voor Q^- hebben we het volgende resultaat: er is een rij $\{\epsilon_p\}_{p=1}^{\infty}, \epsilon_p > 0, p = 1, 2, \ldots, \epsilon_p \downarrow 0$ als $p \to \infty$ zodanig dat voor iedere ϵ_p zuiver imaginaire enkelvoudige eigenwaarden $\lambda_k(\epsilon_p), 1 \leq k \leq n_p$ bestaan van het bijbehorende eigenwaarde probleem voor Q^-. Bovendien geldt: $\lim_{p \to \infty} n_p = +\infty$. Een quantum effect treedt op, waarbij de quantisatie wordt uitgedrukt door de kleine parameter ϵ. In deze zin is de operator P^+ (voor $x_0 < 0$) dispersief. Tot slot wordt in hoofdstuk 5 een specifieke differentiaal-eindige differentie benadering van

de lineaire hyperbolische operator $\partial_{x_0} + \partial_{x_1}$ bestudeerd. De singulariteiten van Green- en Poissonkernen worden onderzocht en eigenwaarden van het corresponderende eigenwaarde probleem worden bepaald.

Curriculum Vitae

Ik ben geboren op 26 mei 1964 in Warnsveld. Van 1976 tot 1982 heb ik het Stedelijk Lyceum in Zutphen bezocht, alwaar ik het VWO-diploma behaalde. Daarna ben ik aan de Katholieke Universiteit Nijmegen wiskunde gaan studeren. In 1987 slaagde ik voor het doctoraalexamen en behaalde tevens een eerstegraads lesbevoegdheid. Vanaf 1987 tot 1991 ben ik assistent in opleiding geweest aan dezelfde universiteit. In deze periode verrichtte ik onderzoek op het gebied van de singuliere storingen onder leiding van Prof. Dr. L.S. Frank en assisteerde ik hem bij het geven van werk- en hoorcolleges. De meeste resultaten van het onderzoek zijn weergegeven in dit proefschrift. Vanaf november 1991 vervul ik mijn vervangende dienstplicht aan de Katholieke Universiteit Brabant in Tilburg, alwaar ik assisteer bij het geven van wiskunde-onderwijs aan studenten economie en econometrie.